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Quantum Field Theory III

Exercise sheet 5

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5. Effective field theory of the disordered metal

This exercise is devoted to the derivation of the σ-model of the disordered metal. We consider the free fermions in the dimension d described by the Hamiltonian $\hat{H} = -\frac{\nabla^2}{2m} + V(\mathbf{x}) = \hat{H}_0 + V(\mathbf{x})$, where $V(\mathbf{x})$ is the Gaussian correlated random potential. Its strength γ is fixed by the correlator $\langle V(\mathbf{x})V(\mathbf{x}')\rangle = \gamma\delta(\mathbf{x}-\mathbf{x}')$. As was shown on the lecture the R-times replicated partition sum, when averaged over the disorder, assumes the form

$$\langle Z^R \rangle = \int \mathcal{D}(\psi,\bar{\psi}) e^{-S_0[\psi,\bar{\psi}]-S_{\text{dis}}[\psi,\bar{\psi}]}, \tag{1}$$

where $S_0 = -\sum_{s,a}\int d\mathbf{x}\,\bar{\psi}^a_s(\epsilon + i(-1)^s\delta - \hat{H}_0)\psi^a_s$ is the free action of the system defined in terms of the Grassmann fields ψ^a_s and $\bar{\psi}^a_s$ with $a = 1,\dots R$ being the index in the replica space and $s = \pm$ discriminating between the retarded/advanced (R/A) sectors, while $S_{\text{dis}} = \frac{\gamma}{2}\sum_{ab,ss'}\int d\mathbf{x}\,\bar{\psi}^a_s\psi^a_s\bar{\psi}^b_{s'}\psi^b_{s'}$ is an effective two-fermion interaction due to the random potential. By decoupling this interaction term in the particle-hole channel with the help of the Hermitian $2R\times 2R$ matrix $B = B^\dagger$ (with elements $B^{ab}_{ss'}(\mathbf{x})$ from replica and R/A spaces) and integrating out the fermion fields, the partition sum (1) is reduced to

$$\langle Z^R \rangle = \int \mathcal{D}B e^{-S_{\text{eff}}[B]}, \quad S_{\text{eff}}[B] = \frac{1}{2\gamma}\int d\mathbf{x}\,\text{tr}B^2(\mathbf{x}) - \ln\det(\epsilon + i\delta\tau_3 - \hat{H}_0 + iB), \tag{2}$$

where Pauli matrix τ_3 operates in R/A space.

a) Check that the saddle point equation of the action $S_{\text{eff}}[B]$ assumes the form of the 'self-consistent Born approximation'

$$B(\text{x}) = i\gamma\langle\mathbf{x}|(\epsilon + i\delta\tau_3 - \hat{H}_0 + iB)^{-1}|\mathbf{x}\rangle. \tag{3}$$

Then verify that its replica symmetric homogeneous in space solution has the form $\bar{B} = \frac{1}{2\tau}\tau_3\otimes\mathbb{1}^{R\times R}$, where $1/\tau = 2\pi\nu\gamma$ is the scattering rate and ν is the density of states. (**Hint**: Use the momentum space and a simplification $\int d\mathbf{p}f(\epsilon_{\mathbf{p}}) = \nu\int d\xi f(\xi)$ for the momentum integrals if they depend on $\mathbf{p}$ only through the dispersion relation $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$).

b) Persuade yourself that other homogeneous solutions of Eq. (3) are given by $\bar{B} = Q/2\tau$ with $Q = T\tau_3T^{-1}$, where $T\in U(2R)$ is an arbitrary unitary matrix operating in the direct product of (replica) $\otimes$ (R/A) spaces. After that verify the sequence of simple facts: (i) the original symmetry of the model is $G = U(2R)$ in the limit $\delta\to 0^+$; (ii) a trivial saddle point $\bar{B}$ is a spontaneously broken state of the action $S[B]$ with the symmetry $H = U(R)\times U(R)$; (iii) on giving a spatial dependence to rotations $T(\mathbf{x})$, the matrix $Q(\mathbf{x})$ spans the Goldstone manifold G/H of the soft modes in the disordered metal.

c) Assume that $T(\mathbf{x})$ is a slow filed which changes on the scale $L \gg \lambda_F \sim 1/k_F$. Prove that the action (2) evaluated on the Goldstone manifold can be approximated by

$$S[Q] \simeq \ln\det\Big(\epsilon + (i/2\tau)\tau_3 - \hat{H}_0 + (i/m)\hat{\mathbf{p}}\mathbf{A}\Big) + \text{Const}, \tag{4}$$

where $\hat{\mathbf{p}}$ is the momentum operator and $\mathbf{A} = T^{-1}\nabla T$ can be thought of as the emergent non-abelian gauge field.

d) Using the well known relation $\ln\det(...) = \operatorname{tr}\ln(...)$ expand $S[Q]$ up to the second order in $\mathbf{A}$. Check that in linear order you get zero, while the 2nd order term is simplified to

$$S^{(2)}[Q] = -\sigma \int d\mathbf{x} \sum_{\mu=1}^{d} \operatorname{tr}\left(P_+ A_\mu P_- A_\mu\right) = \frac{\sigma}{8} \int d\mathbf{x} \sum_{\mu=1}^{d} \operatorname{tr}(\partial_\mu Q)^2. \tag{5}$$

Here $P_\pm = (1 \pm \tau_3)/2$ are projectors on the retarded/advanced subspaces and $\sigma = 2\pi\nu D$ is the conductivity expressed in terms of the diffusion coefficient $D = v_F^2\tau/d$ (**Hint**: Use the momentum shell integration justified in the weak disorder limit $1/\tau \ll \epsilon_F$. For that you can approximate a momentum by $\mathbf{p} \simeq p_F\mathbf{n}$ with $\mathbf{n}$ being a unit vector on the Fermi sphere and then reduce momentum integrals to those over $\mathbf{n}$ and the energy variable ξ).

References

[1] Altland, A. and Simons, B. *Condensed Matter Field Theory*, 2nd edition, Cambridge University Press (2010)

[2] A. Kamenev, *Field Theory of Non-Equilibrium systems* (Cambridge University Press, 2011)