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Quantum Field Theory III

Exercise sheet 6

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6. θ-terms in dimensions $d = 2$ & 3

The aim of this exercise is to elaborate in details the construction of θ-terms using two simple examples.

a) Let the target manifold T be a unit 2-sphere ($T = S^2$) which has the natural embedding into $\mathbb{R}^3$ given by a unit vector $\mathbf{n}$: $S^2 = \{\mathbf{n} \in \mathbb{R}^3 | \mathbf{n}^2 = 1\}$. The Cartesian coordinates of $\mathbf{n} = (n^1, n^2, n^3)$ are expressed via the spherical ones in the standard manner:

$$n^1 = \sin\theta\cos\phi, \quad n^2 = \sin\theta\sin\phi, \quad n^3 = \cos\theta. \tag{1}$$

Consider now the 2-from $\omega = \frac{1}{2}\epsilon^{ijk} n^i dn^j \wedge dn^k$ on S^2. Using relations (1) check that ω is reduced to the volume element on the sphere, i.e. $\omega = \sin\theta \mathrm{d}\theta \wedge d\phi$ with $\int_T \omega = 4\pi$.

b) Let $Q = g\hat{\tau}_3 g^{-1}$ is some order parameter field with $g = e^{-i\phi\hat{\tau}_3/2} e^{-i\theta\hat{\tau}_2/2} e^{-i\psi\hat{\tau}_3/2}$ being the Euler parametrization of the group element in SU(2) and $\hat{\tau}_k$ are Pauli matrices. Verify that

$$Q = n^k(\theta, \phi)\hat{\tau}_k = \mathbf{n} \cdot \hat{\tau} \tag{2}$$

and angle ψ does not contribute. This identity means that all matrices Q form the coset space SU(2)/U(1) $\simeq S^2$ being isomorphic to sphere.

c) Using relation (2) check that the volume 2-form ω on S^2 admits the representation

$$\omega = \frac{1}{4}\mathrm{tr}(QdQ \wedge dQ). \tag{3}$$

d) Consider now the mapping $\mathbf{n} : (x, y) \to \mathbf{n}(x, y)$ from the physical base manifold $M = \mathbb{R}^2$ onto S^2 [not to be confused with the mapping (1) discussed above]. Show that

$$\int_M \mathbf{n}^*\omega = \int_M \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_y \mathbf{n}) dx \wedge dy = \frac{1}{2}\int_M \mathrm{tr}(Q\partial_x Q \partial_y Q) dx \wedge dy. \tag{4}$$

e) Let now the target manifold be the group $T = \mathrm{SU}(2)$, which element in the Euler coordinates is given by the 2×2 matrix $g(\theta, \phi, \psi)$, where $0 \le \psi < 2\pi$, $0 \le \theta \le \pi$ and $0 \le \phi < 4\pi$. Consider the 3-form on T:

$$\omega_3 = \frac{1}{24\pi^2}\mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) \tag{5}$$

Verify that it equals to the volume element on SU(2), i.e. $\omega_3 = \sin\theta \mathrm{d}\psi \wedge d\theta \wedge d\phi/(16\pi^2)$ with $\int_{\mathrm{SU}(2)} \omega_3 = 1$.

f) Consider now the mapping $g : \mathbb{R}^3 \to \mathrm{SU}(2)$ from the physical base manifold $M = \mathbb{R}^3$ onto the target group manifold $T = \mathrm{SU}(2)$. Check that for pullback from ω_3 one has

$$\int_M \mathbf{g}^* \omega_3 = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \epsilon^{\mu\nu\rho} \mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g) dx^1 dx^2 dx^3 \tag{6}$$

References

[1] Altland, A. and Simons, B. *Condensed Matter Field Theory*, 2nd edition, Cambridge University Press (2010)