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Quantum Field Theory III

Exercise sheet 7

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7. Equation of motion for the WZW action

Consider the non-linear σ-model action in $d = 2$ dimensions

$$S_{\rm WZW}[g] = S_2[g] + \Gamma[g] = \frac{1}{8\pi}\int_{S^2} d^2x\,\mathrm{tr}(\partial_\mu g \partial_\mu g^{-1}) + \Gamma[g], \tag{1}$$

where the field $g \in \mathrm{U}(N)$, the two-dimensional space is compactified to the 2-sphere S^2, and $\Gamma[g]$ is the Wess-Zumino-Witten term

$$\Gamma[g] = -\frac{i}{12\pi}\int_{B^3} d^3x\,\epsilon^{\mu\nu\rho}\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\nu g). \tag{2}$$

It is assumed here that the sphere S^2 serves as the boundary of one half of a three-dimensional unit ball B^3 (i.e. $\partial B^3 \equiv S^2$) and the field $g(x_0, x_1)$ initially defined in $d = 2$ is smoothly extended in the third (x_2)-direction.

a) Check first that without the WZW term, the saddle point equation corresponding to the σ-model action $S_2[g]$ has the form $\partial_\mu(g^{-1}\partial_\mu g) = 0$.

b) Derive further the saddle point equation of the full action $S[g]$ with the WZW term included. You have to arrive to the result $\partial_z \bar{j} = 0$, where the current $\bar{j} \propto (\partial_{\bar{z}} g)g^{-1}$ and the complex coordinates are defined as $z = (x_0 + ix_1)/\sqrt{2}$ and $\bar{z} = (x_0 - ix_1)/\sqrt{2}$.

8. The σ-model action of the disordered BCS superconductor

The disordered s-wave BCS superconductor (in dimension $d = 2, 3$) can be described by the following low-energy action

$$S[Q] = \frac{1}{2}\pi\nu \int d^d\mathbf{r}\,\mathrm{tr}\left[\frac{1}{4}D(\nabla_{\mathbf{r}}Q)^2 - \epsilon_n\tau_3\,Q - \Delta\tau_2\,Q\right], \tag{3}$$

where the field Q is the Matsubara Green's function at energy ϵ_n (evaluating at coinciding spatial points) obeying the constrain $Q^2 = 1$, Pauli matrices τ_i act in the Nambu space, Δ is the BCS gap, D is the diffusion coefficient, and ν is the density of states (DoS) in the normal state at Fermi energy.

a) By minimazing the action $S[Q]$ derive the saddle point equation for Q,

$$D\nabla_{\mathbf{r}}(Q\nabla_{\mathbf{r}}Q) = [\epsilon_n\tau_3 + \Delta\tau_2, Q]\,, \tag{4}$$

which is known as the Usadel equation.

b) Using the above result, find the stationary (position-independent) Green's function $Q(\epsilon_n)$. Performing the analytic continuation to the real energy, deduce the DoS in the disordered superconductor, $\nu(\epsilon) = \nu\mathrm{Re}\,\mathrm{tr}\,\tau_3 Q(\epsilon_n)\Big|_{i\epsilon_n\to\epsilon+i0^+}$. Compare it with the density of states known from the BCS model.

References

[1] Altland, A. and Simons, B. *Condensed Matter Field Theory*, 2nd edition, Cambridge University Press (2010)