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Quantum Field Theory III

Exercise sheet 8

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9. Non-Abelian gauge fields

Let 1-form $\mathcal{A} = A_\mu \, dx^\mu$ (defined on a physical space-time manifold $\mathbb{R}^4$) be the non-Abelian gauge field with A_μ taking the values in Lie algebra of the non-commutative Lie group. Then the 2-form of the field strength tensor is defined by $\mathcal{F} = d\mathcal{A} - ig\mathcal{A} \wedge \mathcal{A}$ and g is the coupling constant ("charge"). When written in terms of coordinates, one has

$$\mathcal{F} = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{1}$$

For any form ω on $\mathbb{R}^4$ with values in the corresponding Lie algebra one can define the *covariant* exterior derivative by the relation $D\omega \equiv d\omega - ig[\mathcal{A}, \omega]$, where $[\mathcal{A}, \omega] = \mathcal{A} \wedge \omega - \omega \wedge \mathcal{A}$.

a) Prove the Bianchi identity, $D\mathcal{F} = 0$.

b) Check that in the coordinates the Bianchi identity assumes the form

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0, \qquad \mu \neq \nu \neq \lambda, \tag{2}$$

where the covariant derivative is defined by $D_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} - ig[A_\mu, F_{\nu\lambda}]$.

c) Verify that for $U(1)$ electromagnetic gauge field the Bianchi identity corresponds to two Maxwell equations, $\mathrm{div}\mathbf{B} = 0$ and $\mathrm{rot}\,\mathbf{E} = -\partial_t \mathbf{B}$.

d) Consider now the Yang-Mills action $S[A_\mu] = -\frac{1}{8}\int d^4x \, \mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$. Using the least action principle check that the classical equations of motion for the Yang-Mills fields have the form

$$D_\mu F^{\mu\nu} = 0. \tag{3}$$

Also check that in the Abelian case they are reduced to other two Maxwell equations $\mathrm{div}\,\mathbf{E} = 0$ and $\mathrm{rot}\,\mathbf{B} = \partial_t \mathbf{E}$.