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Quantum Field Theory III

Exercise sheet 9

This exercise will be discussed on 14.06.2016

10. Non-Abelian Chern-Simons action

Consider the model of massive Dirac fermions in dimension $d = 3 + 0$ interacting with the non-Abelian gauge field A_μ defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu + m)\psi, \tag{1}$$

where the matrices $\gamma_\mu \equiv \sigma_\mu$ form the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$ in $d = 3$ and A_μ is an element of non-Abelian Lie algebra (for instance in the case of SU(2) algebra $A_\mu = \frac{1}{2}\sum_{k=1}^{3} A_\mu^k \tau_k$, where the Pauli matrices τ_k operate in the $n = 2$ dimensional color/flavor space).

The goal of the exercise is to derive the effective action of the theory

$$S_{\text{eff}}[A] = -\text{tr}\ln(\partial\!\!\!/ + m - iA\!\!\!/) = \text{Const} + \frac{1}{2}\text{tr}(GA\!\!\!/)^2 + \frac{1}{3}\text{tr}(GA\!\!\!/)^3 + \ldots \tag{2}$$

in the long wave limit. Here $A\!\!\!/ = A_\mu\gamma^\mu$ and $G^{-1} = -i\partial\!\!\!/ - im$ is the inverse electron propagator. Let $\Lambda \gg m$ is the high-momentum cut-off.

a) Check that the 2nd–order term of the effective action, when written in the momentum space, takes the form

$$S^{[2]}_{\text{eff}}[A] = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\text{tr}\Big(A_\mu(-q)A_\nu(q)\Big)\Pi^{\mu\nu}(q), \tag{3}$$

where the polarization operator reads

$$\Pi^{\mu\nu}(q)\Big|_{q\ll m\ll\Lambda} = \frac{\Lambda}{3\pi^2}\delta^{\mu\nu} + \frac{\text{sgn}(m)}{4\pi}\epsilon^{\mu\nu\lambda}q_\lambda \tag{4}$$

b) Similarly, in the 3d order you have to find that

$$S^{[3]}_{\text{eff}}[A] = \frac{1}{2}\int \frac{d^3q\, d^3p}{(2\pi)^6}\text{tr}\Big(A_\mu(-q-k)A_\nu(q)A_\lambda(k)\Big)\Pi^{\mu\nu\lambda}(q,k) \tag{5}$$

with

$$\Pi^{\mu\nu\lambda}(q,k)\Big|_{(q,k)\ll m} = -i\frac{\text{sgn}(m)}{4\pi}\epsilon^{\mu\nu\lambda} \tag{6}$$

c) With the above results at hand verify that the effective action takes a local form in the real space,

$$S_{\text{eff}}[A] = \frac{\Lambda}{6\pi^2} \int d^3x \text{tr}(A_\mu A_\mu) - \pi \text{sgn}(m) S_{\text{CS}}[A], \tag{7}$$

where $S_{\text{CS}}[A]$ is the non-Abelian Chern-Simons action

$$S_{\text{CS}}[A] = \frac{i}{8\pi^2} \int \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{8}$$