Non-abelian gauge fields

- Dirac Lagrangian

ⓐ $\mathcal{L} = \bar{\psi}\, i\gamma^\mu (\partial_\mu + ie A_\mu)\psi$; $\bar{\psi} = \psi^\dagger \gamma^0$

$\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$ → form Clifford algebra

$\mu = 0, 1, \dots, d-1$ & imaginary time $\underline{d=4}$

ⓑ $\nabla_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$ → covariant derivative

ⓒ $\mathcal{L}$ is gauge invariant :

Let $\psi' = e^{i\chi}\psi$; $A_\mu' = A - \frac{1}{e}\partial_\mu \chi$; $\chi(x) \in \mathbb{R}$ (real)

then $\mathcal{L}[\psi', A'] = \mathcal{L}[A, \psi]$

- Quantum electrodynamics (abelian $U(1)$ gauge theory)

Action: $S = \int \mathcal{L}\, d^4x - \frac{1}{4}\int d^4x\, F^{\mu\nu} F_{\mu\nu}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ → e.m. field tensor

I Geometry of Yang-Mills fields

Given : $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$ → n-component spinor field

(jargon: it has colour / flavour)

- Q: how to use colour structure to define gauge structure? $U(1)$ phase → too small

- A: Consider n-dimensional representation of non-Abelian Lie algebra $\mathfrak{g}$ in the colour space.

(A) Basics of Lie groups & algebras

- $G = SU(2) \to$ simplest example

$$G = \{ g = \exp[\tfrac{i}{2} \alpha^a \hat{\tau}_a] \mid \alpha^a \in R \} \quad ; \quad a = 1, 2, 3$$

$$[\tfrac{1}{2}\hat{\tau}_a, \tfrac{1}{2}\hat{\tau}_b] = \tfrac{i}{2} \varepsilon_{abc} \hat{\tau}_c \quad , \quad \tau_a \to \text{Pauli matrices}$$

$\dim G = 3 \quad , \quad n = 2$

- General case: let G be compact (semi simple) Lie group
$\mathfrak{g} \to$ its algebra , $\dim(\mathfrak{g}) = k$

(a) Def: (i) The set $\{ \hat{T}_a^+ = \hat{T}^a \ , \ a = 1, \dots, k \}$
is basis of $\mathfrak{g}$

(ii) $(\hat{T}_a)^{n \times n} \to$ Hermitian matrix , $n \to$ defines representation

(iii) $[\hat{T}_a, \hat{T}_b] = i f_{abc} \hat{T}_c \to$ comutation relations
$f_{abc} \to$ "structure constants" (do not vary with n)

(iv) $\hat{T}_a$ obey Jacobi identity

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0$$

(b) Lie group

$$G = \{ g = \exp[i \sum_{a=1}^{k} \alpha^a \hat{T}_a] \mid \alpha^a \in \mathbb{R} \}$$

i.e. exponentiation of algebra

! Set $\{\alpha^a\}$ parametrizes all possible g.

(B) Yang – Mills construction:

$S^{\dagger} = S^{-1}$

(a) $\mathcal{L} = \bar{\psi}\, i \gamma^{\mu} \partial_{\mu} \psi \;, \quad \psi' = \hat{S} \psi \;, \quad \hat{S} \in G \subset U(n)$

global symmetry

$\Rightarrow$ make it local !

(b) (i) $\psi' = \hat{S}(x) \psi$ with $\hat{S}(x) = \exp[i \alpha^{a}(x) \hat{T}_a]$.

(ii) Gauge field: $\hat{A}_{\mu}(x) = A^{a}_{\mu}(x) \hat{T}_a \in \mathfrak{g}$

takes values on Lie algebra

(iii) Covariant derivative is defined by

$$\nabla_{\mu} \psi \equiv \partial_{\mu} \psi - i g \hat{A}_{\mu} \psi \quad | \quad g \rightarrow \text{coupling const.}$$

(iv) Under gauge transform $\hat{A}_{\mu}$ changes as

$$A'_{\mu} = S A_{\mu} S^{-1} - \frac{i}{g} (\partial_{\mu} S) S^{-1} \qquad (1)$$

(c) <u>Lemma:</u> $\nabla'_{\mu} \psi' = S(x) \nabla_{\mu} \psi$

<u>Proof:</u>

$$\nabla'_{\mu} \psi' = (\partial_{\mu} - i g A'_{\mu}) S \psi = S(\partial_{\mu} - i g S^{-1} A'_{\mu} S + S^{-1} \partial_{\mu} S) \psi$$

$$= S(\partial_{\mu} - i g A_{\mu}) \psi$$

$\Rightarrow$ we have to <u>demand</u> that

$$-i g A_{\mu} = -i g S^{-1} A'_{\mu} S + S^{-1} \partial_{\mu} S \quad \Rightarrow (1) \quad \blacksquare$$

(d) <u>Result:</u> $\mathcal{L} = \bar{\psi}\, i \gamma^{\mu} \nabla_{\mu} \psi \rightarrow$ gaug. inv. Lagrangian

$$\mathcal{L}[\psi, A_{\mu}] = \mathcal{L}[\psi', A'_{\mu}].$$

ⓔ Field strength tensor: $F_{\mu\nu}$

non-Abelian part

Def: $$F_{\mu\nu} = \frac{i}{g}\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu \underbrace{- ig\left[A_\mu, A_\nu\right]}.$$

$F_{\mu\nu} \in \mathfrak{g}$ → belongs to Lie algebra

If one writes $F_{\mu\nu} = \frac{1}{2} F_{\mu\nu}^{\ a}\, \hat{T}_a$ ⇒

$$F_{\mu\nu}^{\ a} = \partial_\mu A_\nu^{\ a} - \partial_\nu A_\mu^{\ a} + g\, f^{abc} A_\mu^{\ b} A_\nu^{\ c}$$

Lemma: $\underset{\text{simple transf. law!}}{F'_{\mu\nu} = S F_{\mu\nu} S^{-1}}$, since $\hat{\nabla}'_\mu = S \hat{\nabla}_\mu S^{-1}$

ⓕ Yang-Mills Lagrangian:

$$\boxed{\mathcal{L}_{YM} = \bar{\Psi}\, i \gamma^\mu \nabla_\mu \Psi - \frac{1}{8}\, \mathrm{tr}\left(F^{\mu\nu} F_{\mu\nu}\right)}$$

Note: one usually chooses the basis $\hat{T}_a$ such that

$$\mathrm{tr}\left(\hat{T}_a \hat{T}_b\right) = 2\delta^{ab} \quad (\text{"orthogonal basis"})$$

$$\Rightarrow \quad \frac{1}{8}\, \mathrm{tr}\left(F^{\mu\nu} F_{\mu\nu}\right) = \frac{1}{4} \sum_{a=1}^{k} F^{\mu\nu}_{\ a} F_{\mu\nu}^{\ a}$$

gauge inv. construction ⇒

$\mathcal{L}_{YM}$ is locally gauge inv. model!

ⓖ Phys. applications:

ⓐ "Chromodynamics": $G = SU(3)$

ⓑ Weinberg-Salam model of electroweak interactions:
$G = SU(2) \otimes U(1)$

(c) Language of 1 & 2 - forms: (diff. geometry)

1. $\mathcal{A} = \mathcal{A}_\mu dx^\mu \rightarrow$ 1-form on $\mathbb{R}^d$

2. $\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \wedge dx^\nu \rightarrow$ 2-form.

3. Lemma: $\mathcal{F} = d\mathcal{A} - ig\, A \wedge A$

Proof: $d\mathcal{A} = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu \underset{\text{antisymmetrize!}}{=} \frac{1}{2} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \times dx^\mu \wedge dx^\nu$

$-ig\, A \wedge A = -ig \, A_\mu A_\nu \, dx^\mu \wedge dx^\nu =$

$= -\frac{ig}{2} \left[A_\mu, A_\nu \right] dx^\mu \wedge dx^\nu$. ▨

4. Bianchi identity:

(a) $D\mathcal{F} := d\mathcal{F} - ig [\mathcal{A}, \mathcal{F}] = 0$ (exercise (!)) (2)

Here $[\mathcal{A}, \mathcal{F}] \equiv \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A}$.

(b) In the coordinate language it has the form:

$$D_\mu \mathcal{F}_{\lambda\rho} + D_\lambda \mathcal{F}_{\rho\mu} + D_\rho \mathcal{F}_{\mu\nu} = 0, \quad \mu \neq \lambda \neq \rho \qquad (3)$$

where $D_\mu \mathcal{F}_{\lambda\rho} := \partial_\mu \mathcal{F}_{\lambda\rho} - ig \left[A_\mu, \mathcal{F}_{\lambda\rho} \right]$

commutator in Lie algebra

(c) Define dual field tensor:

$${}^*\mathcal{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \mathcal{F}^{\lambda\rho} \Rightarrow$$

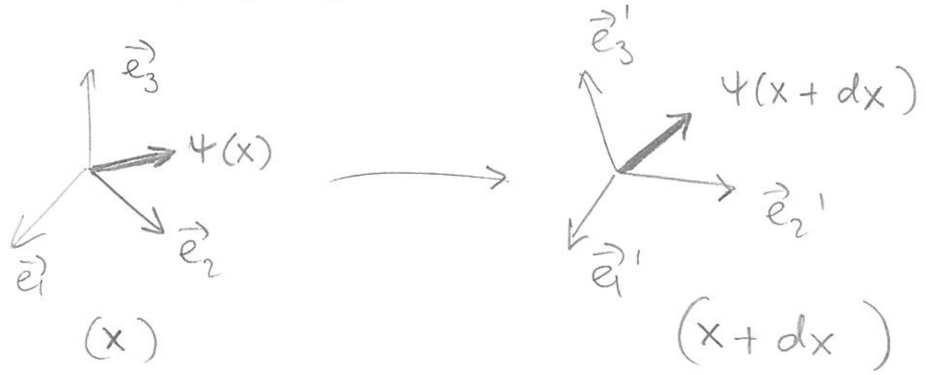
$$D_\mu {}^*\mathcal{F}^{\mu\nu} = \partial_\mu {}^*\mathcal{F}^{\mu\nu} - ig \left[A_\mu, {}^*\mathcal{F}^{\mu\nu} \right] = 0$$

another form of (2) & (3)

(D) Geometrical meaning of vector potential $\mathcal{A}_\mu$

$\psi(x) = \psi^k(x)\, \vec{e}_k(x)$, where

$\{ \vec{e}_k(x) \mid k = 1, \dots, n \}$ → local basie in colour space.



- $d\psi(x) \overset{!}{=} d\psi^k(x)\, \vec{e}_k(x) + \psi^k(x)\, d\vec{e}_k(x)$

  Basis $\vec{e}_k(x)$ may rotate from point to point

- Let us set $d\vec{e}_k(x) = ig\, \vec{e}_\ell\, \mathcal{A}^{\ell}_{\;k,\mu}\, dx^\mu$

  Matrix $\hat{\mathcal{A}}_\mu$ with elements $(\hat{\mathcal{A}}_\mu)^\ell_{\;k} \equiv \mathcal{A}^{\ell}_{\;k,\mu}$

  defines the connection ⇒ specifies basis' rotations.

- We define

$$\delta\psi := \psi^k(x)\, d\vec{e}_k(x) = ig \left(A^{\ell}_{\;k,\mu}\, \psi^k\, \vec{e}_\ell \right) dx^\mu =$$

$$= (\delta\psi)_\mu\, dx^\mu \quad , \text{ where } \quad (\delta\vec{\psi})_\mu = ig\, \hat{A}_\mu\, \psi$$

- Covariant derivative is defined by

$$D\psi = (\psi + d\psi) - (\psi + \delta\psi) = d\psi - ig\, \hat{A}_\mu\, \psi\, dx^\mu$$

$$\Rightarrow \quad \nabla_\mu \psi = \frac{D\psi}{dx^\mu} \overset{!}{=} \partial_\mu \psi - ig\, \hat{\mathcal{A}}_\mu\, \psi . \quad ▨$$