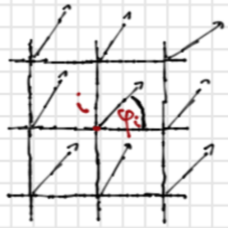
The Berezinsky-Kosterlitz-Thouless (BKT) transition

• Consider classical xy-model in 2d



$$Z = \int \underbrace{\mathcal{D}\varphi}_{\prod_i d\varphi_i} \exp\Big(-\mathcal{J} \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j)\Big) \qquad \mathcal{J} = \overset{\text{ferro. exchange}}{\mathcal{J}_f}/T > 0$$

• Mermin-Wagner Thm.: no symmetry breaking phase transition

... and yet the model has a phase transition

• Consider spin-spin correlation function $C_{i,j} = \langle e^{i\varphi_i} e^{-i\varphi_j} \rangle$

1) Low T: $\mathcal{J} \gg 1$

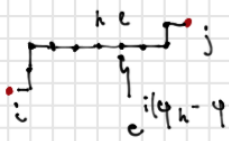
$$S[\varphi] = \mathcal{J} \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j) \simeq -\frac{\mathcal{J}}{2} \int d^2x \, \nabla\varphi \nabla\varphi + \text{const.}$$

$$\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle = \ln\big(|x_i - x_j| / \underset{\text{lattice spacing}}{a}\big) / \pi \mathcal{J}$$

$$C_{i,j} = e^{-\frac{1}{2}\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle} = \left(\frac{a}{|x_i - x_j|}\right)^{1/2\pi\mathcal{J}}$$

$\sim$ power law correlations

2) High T, $\mathcal{J} \ll 1$ ~ Expand action in $\mathcal{J}$



$e^{i(\varphi_h - \varphi_l)}$

$$C_{i,j} \sim \mathcal{J}^{\overset{d_{i,j}}{\vdots}} \sim e^{-\ln\mathcal{J} \cdot c \cdot |x_i - x_j|}$$

$d_{i,j}$: Manhattan metric

exponentially decaying cor.

Conclusion: There must be a finite T phase transition. What discriminates 1) from 2)?: 2) accounts for phase windings of compact var. φ.

Strategy: Teach 1) to include windings

Vortices
- cannot be individually destroyed, however
- vortices ('charges') and anti-vortices (anti-charges) may annihilate each other.
- Individual vortex at r_0 described by, e.g., $\varphi(r) = \tan^{-1}\left(\frac{(r - r_0)_2}{(r - r_0)_1}\right) + \frac{\pi}{2}$ + small fluctuations $(|r - r_0| \gtrsim a)$

a vortex

a vortex/anti-vortex pair

• have action cost ($r_0 = 0$): • $\nabla\varphi = \frac{1}{r}\,\underline{e}_\varphi$

$$S_v = \frac{\mathfrak{Z}}{2} \int_a^L r\,dr \int_0^{2\pi} d\varphi \, \frac{1}{r^2} + \underbrace{S^{uv}(a)}_{\text{short distance action}} =$$

$$= \pi \mathfrak{Z} \ln\left(\frac{L}{a}\right) + S^{uv}(a)$$

$\rightsquigarrow$ A single vortex costs action $\sim \ln L$

• Q: Will vortices be present in the system?

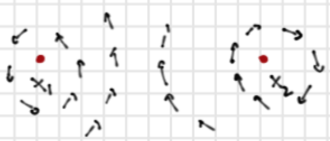
Free energy of a vortex: $F_V = -T \ln Z_V = -T \ln e^{-S_V} \times \underbrace{\left(\frac{L}{a}\right)^2}_{\#\text{ of different vortex center coord.}}$

$$= \ln\left(\frac{L}{a}\right) \times \{ \underbrace{\pi \mathfrak{Z}_f}_{\text{vortex energy}} - \underbrace{T\,2}_{\text{entropy}} \} + \text{const.}$$

A: For $T \geq T_{BKT} = \frac{\pi}{2} \mathfrak{Z}_f$ vortex formation becomes favorable.

• Q: What happens in high temperature phase?

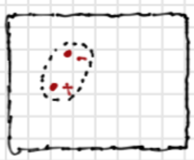
A: Estimate action of vortex anti-vortex pair



$$S(+, x_1; -, x_2) = 2 S^{uv}(a) + 2\pi \mathfrak{Z} \ln\left(|x_1 - x_2| / a\right)$$

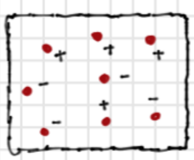
cf. action of two particles in 2d with electric charge $\pm q(a) = \pm 2\pi\mathfrak{Z}$ and fugacity $y(a) = \exp(-S^{uv}(a))$

1) low $T < T_{BKT}$



thermal fluctuations: tightly bound dipoles

2) high $T > T_{BKT}$



plasma of charged particles

RG - Analysis of BKT - Transition

Picture: Low T: $\underset{r}{\oplus} \leftarrow \text{log-int} \rightarrow \underset{r'}{\ominus}$

High T: $\oplus \quad \ominus \quad \oplus \quad \ominus \quad \oplus \quad \ominus \quad \oplus \quad \ominus$

screening $\rightarrow$ diminished interaction

Effective vortex partition sum:

$$Z_+ = \sum_{N=0}^{\infty} \frac{y_0^{2N}}{(N!)^2} \prod_{i=1}^{N} \int d^2 r_i \; e^{4\pi^2 J \sum_{i<j} q_i q_j C(r_i - r_j)}$$

y_0: vortex fugacity

q_i: vortex charge ± 1 $\left(\sum_i q_i = 0 \right)$

$C(x) = \ln|x| / 2\pi$ 2d - Coulomb int.

Explore screening perturbatively in y_0: effective interaction:

$$\underset{r}{\oplus} - \underset{r'}{\ominus} \; + \; \underset{r}{\oplus} = \underset{s}{\ominus} \underset{s'}{\oplus} = \underset{r'}{\ominus} \; + \; \mathcal{O}(y_0^4)$$

$$e^{-J_{eff}(r-r')} = \left\langle e^{-4\pi^2 J \, C(r-r')} \right\rangle_+ =$$

$$= \left(\oplus - \ominus + y_0^2 \int ds\, ds' \; \underset{r}{\oplus} = \underset{s}{\ominus} \underset{s'}{\oplus} = \underset{r'}{\ominus} \right) \left(1 + y_0^2 \int ds\, ds' \; \underset{s}{\oplus} - \underset{s'}{\ominus} \right)^{-1}$$

$$\Big[\; \underset{s}{\oplus} - \underset{s'}{\ominus} = e^{-4\pi^2 J \, C(s-s')}$$

⋮ some technical work

⋮ (cf. Altland & Simons)

⋮

$$\leadsto \quad J_{eff}(v-v') = 4\pi^2 \mathfrak{z}_{eff}\, C(v-v')$$

$$\mathfrak{z}_{eff} = \mathfrak{z} - 4\pi^3 \mathfrak{z}^2 y_0^2 \int_1^\infty \underbrace{dx}_{\text{length in unit of cutoff}}\, x^{3-2\pi\mathfrak{z}} + O(y_0^4) = \mathfrak{z} - \mathfrak{z}^2 \circledcirc$$

For $\mathfrak{z} < \frac{2}{\pi}$: divergence at large x ! (Vortex unbinding) Expect feedback into $y \leadsto$ Coupled flow of $(\mathfrak{z}^{-1}, y)$

Quantitative description: successive integration over short distances removing charges.

$$\mathfrak{z}_{eff} = \mathfrak{z} - \mathfrak{z}^2 \circledcirc \leadsto \mathfrak{z}_{eff}^{-1} = \mathfrak{z}^{-1} + \circledcirc = \underbrace{\mathfrak{z}^{-1} + \circledcirc_1^b}_{\tilde{\mathfrak{z}}^{-1}} + \circledcirc_b^\infty$$

$$\tilde{\mathfrak{z}}^{-1} = \mathfrak{z}^{-1} + 4\pi^3 y_0^2 \int_1^b dx\, x^{3-2\pi\mathfrak{z}}$$

$$\mathfrak{z}_{eff}^{-1} = \tilde{\mathfrak{z}}^{-1} + 4\pi^3 \tilde{y}_0^2 \int_1^\infty dx\, x^{3-2\pi\mathfrak{z}} \qquad \tilde{y}_0 = b^{2-\pi\mathfrak{z}}\, y_0$$

Def.: $b = e^l$ and differentiate flow $(\tilde{\mathfrak{z}}^{-1}(l), \tilde{y}_0(l))$ at $l=0$. $\left.\frac{d}{dl}\right|_{l=0} = \left.\frac{d}{db}\right|_{b=1}$

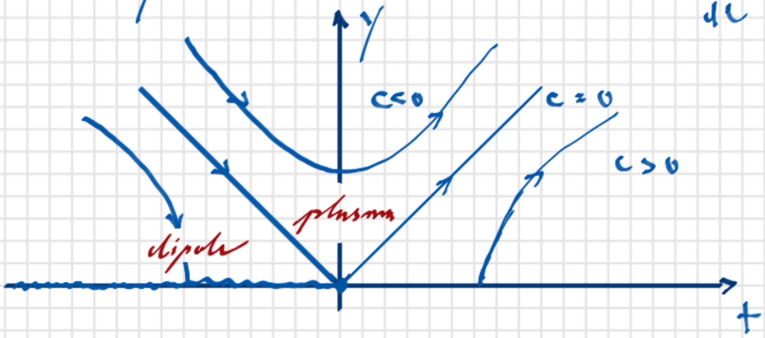
Omit '$\sim$'.

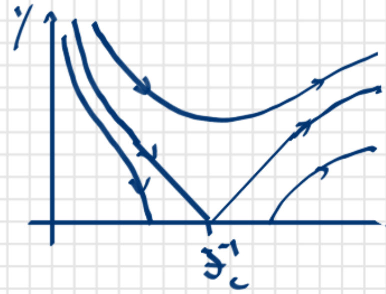
$$\left|\begin{aligned} \frac{d\mathfrak{z}^{-1}}{dl} &= 4\pi^3 y^2 \\ \frac{dy}{dl} &= (2-\pi\mathfrak{z})\, y \end{aligned}\right. \qquad \text{BKT flow equations}$$

Fixed point at $(\mathfrak{z}^{-1}, y) = (\frac{\pi}{2}, 0)$. What happens in vicinity of critical point?

$$\begin{cases} \dfrac{d\zeta^{-1}}{dl} = 4 n^3 y^2 \\[2mm] \dfrac{dy}{dl} = (2 - n\zeta)\, y \end{cases} \quad \rightsquigarrow \quad t = \zeta^{-1} - \frac{n}{2} \quad \begin{cases} \dfrac{dt}{dl} = 4 n^2 y^2 \\[2mm] \dfrac{dy}{dl} = \dfrac{4}{n}\, t y \end{cases}$$

Notice: $t^2 - 4 n y^2 = c$ is conserved! $\dfrac{dc}{dl} = 0$





large scale picture

Path integral for spin (in a nutshell)

Two approaches for constructing spin path integral

1) Trotterization of time evolution in terms of spin coherent states.
2) Quick and dirty based on consistency arguments. (Not to be trusted unless without 1) as backup)

The quick approach:

- Classically: spin is angular momentum of fixed magnitude S.
  How many degrees of freedom? Answer: Only 1! In the phase space of spin we don't find more than one commuting function $\{S_x, S_y\} \neq 0$...
- Need phase space resembling a sphere... The 2-sphere itself is a phase space. (Recall: phase space is a manifold with non-degenerate alternating 2-form, the symplectic form ω. E.g. $\omega = \sum_i dq_i \wedge dp_i$). It's symplectic form is the area 2-form. Local representation: $\omega = \sin\theta \, d\theta \wedge d\varphi = d\varphi \wedge d(\cos\theta)$. Conclusion: $(\varphi, \cos\theta)$ form a conjugate pair of 'coordinate and momentum.)
- For concreteness, assume angular momentum $\underline{S} = \underline{n} S$ in a magnetic field $\underline{B} = \underline{e}_z B$. Hamiltonian $H = -\underline{B} \cdot \underline{S} = -BS\cos\theta$. What is the corresponding Hamiltonian action? Note: $\omega = -d(d\varphi \cos\theta) \sim$,

$$S[\varphi, \cos\theta] = S\int dt \, \dot{\varphi} \cos\theta - SB \int dt \cos\theta$$

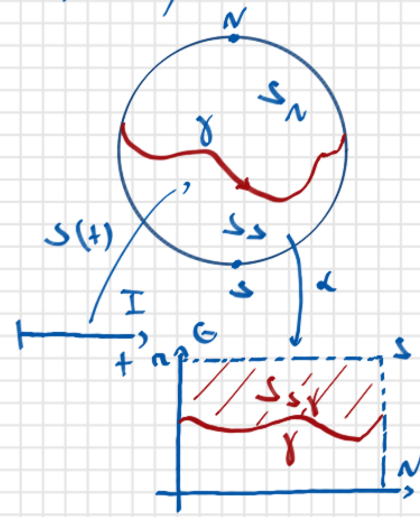
Building faith: Equations of motion

$$\frac{\delta S}{\delta \varphi} = 0 : \quad \theta = \text{const.}$$

$$\frac{\delta S}{\delta \theta} = 0 : \quad -S \sin\theta \, \dot{\varphi} + SB \sin\theta = 0 \iff \dot{\varphi} = B$$

Correct description of precession motion.

• Spin quantization.



Consider canonical piece of action

$$S_c[\gamma] = S\int_I dt\, \dot\varphi \cos\theta$$ Singular at both N and S

Improved representation: $S_c^{N/S} = S\int_I dt\,\dot\varphi(\cos\theta \pm 1)$ added inessential full derivative. (Microscopic construction of PI yields these representations anyway.)

Geometry of $S_c^{N/S}$: Describe curve in coordinate chart which is singular at S or N. $S_c^{N/S}$ is integral of differential 1-form along curve in coordinate domain:

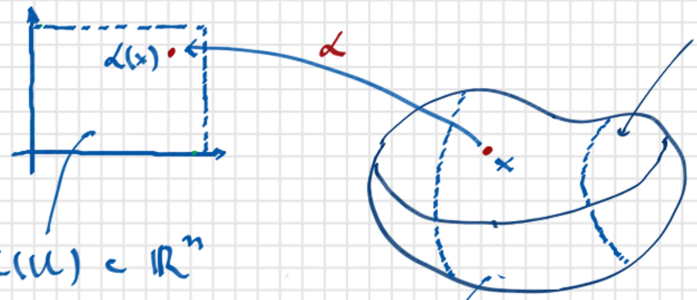
$$S_c^{N/S} = S\int_I dt\, \frac{d\varphi}{dt}(\cos\theta\pm1) = S\int_\gamma d\varphi(\cos\theta\pm1) \underset{\text{Stokes}}{=}$$

$$= -S\int_{S_{N/S,\gamma}} d\varphi\wedge d\cos\theta = +S \times \underbrace{\begin{cases} S_{N,\gamma} \\ -S_{S,\gamma}\end{cases}}_{\equiv \text{ geometric area of } S_{N/S,\gamma}}$$

Ambiguity between the choices: $S(S_{N,\gamma} - (-) S_{S,\gamma}) = 4\pi S$. In path integral $e^{i4\pi S}$ is unobservable, iff $S\in\frac{1}{2}\mathbb{Z}$. Spin quantization!

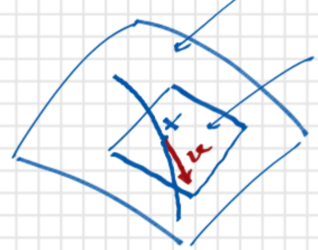
Differential forms in a nutshell

The setting:



Standard notation: $\alpha(x) = (\alpha^1(x), \dots, \alpha^n(x)) \equiv (x^1, \dots, x^n)$ the coordinates of x in the chart (U, α). In a different chart (α', U') with $x \in U \cap U'$ $x \overset{\alpha'}{\mapsto} (x'^1, .., x'^n)$. One point – different coordinates. ('Interesting' manifolds – spheres, tori, group manifolds – require more than one chart to define an atlas $\{(U_i, \alpha_i)\}$ $\bigcup_i U_i = M$.

Tangent space: Locally each Mf looks flat.



Tangent space $T_x M$ is a vector space of Dimension n representing a flat approximation to M at x. In general: $T_x M \neq T_{x'} M$ $x \neq x'$ but $T_x M \cong \mathbb{R}^n$. Elements of $v \in T_x M$ are tangent vectors. Each v corresponds to a curve $\gamma: I \to M$, $t \mapsto \gamma(t)$ $\gamma(0) = x$. v is the tangent vector to γ at $t = 0$. Note: (i) the assignment $\gamma \to v$ is not 1-1 ($\gamma, \gamma' \to v$). (ii) Avoid writing '$d_t \gamma(t)|_{t=0} = v$' because $d_t \gamma$ not defined. The components of v in α are given by

$$v^i = d_t \gamma^i(t)\big|_{t=0}$$

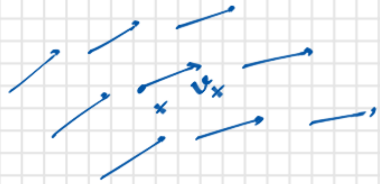
Vectors are differential operators (!) acting on functions $f: M \to \mathbb{R}^d$

$$v(f) = d_t\, f(\gamma(t))$$

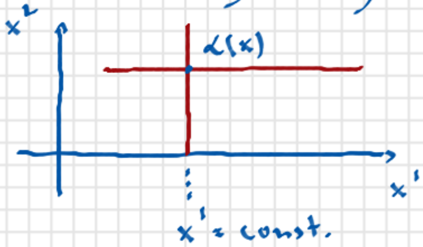
$T_x M = \{ v \mid v \text{ tangent to } M \text{ at } x \}$

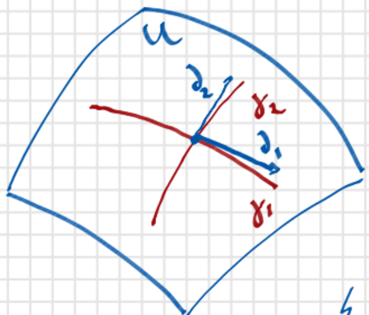
$\bigcup_{x \in M} T_x M \cong TM$ tangent bundle.

A vector field is a smooth map: $v: M \to TM,\ x \mapsto v_x$



Coordinate basis of tangent space. Pick (α, U).

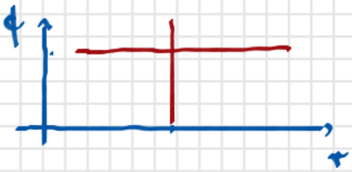


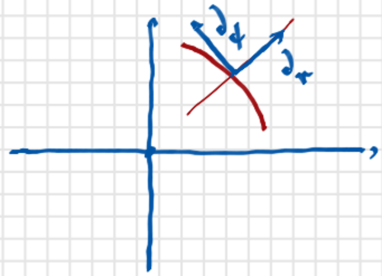


- $\gamma_i(t) = \alpha^{-1}(x^1, \dots, t, \dots, x^n)$
- ∂_i tangent to γ_i. $\partial_i f = d_t\, f(\gamma_i(t)) = \dfrac{\partial f(x)}{\partial x^i}$

f interpreted as function of coords.

Example: $M = \mathbb{R}^2 \setminus \{0\}$, $\alpha = (r, \phi)$





What are cartesian coordinates of ∂_ϕ? $\partial_\phi x^i = d_t\, \gamma_\phi(t)^i = d_t \begin{cases} r\cos(t+\phi) \\ r\sin(t+\phi) \end{cases} =$

$$= \begin{cases} -r\sin\phi \\ r\cos\phi \end{cases}$$

Note: (e_ϕ, e_r) of physics textbooks are normalized $(\partial_\phi, \partial_r)$. Here: No metric.

Note: Coordinate vector fields $\{\partial_i\}$ are basis of TM. Every vector field can be expanded as $v = \sum_i \underbrace{v^i}_{\text{coefficient function}} \partial_i$

Differential 1-Form. A 1-form at $x \in M$, $\phi_x \in T_xM^*$ is an element of the dual space to T_xM, the **cotangent space** T_xM. A **1-form** ϕ on M is defined by a set of forms ϕ_x smoothly depending on x:

$$\phi_x : v_x \mapsto \phi_x(v_x) \in \mathbb{R}$$

$$\phi : v \mapsto \phi(v) \quad \text{a function}$$

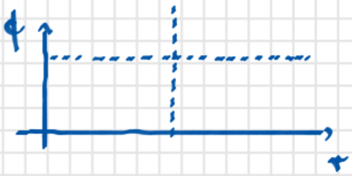
$\phi \in TM^* = \bigcup_x T_xM^*$ the **cotangent bundle**.

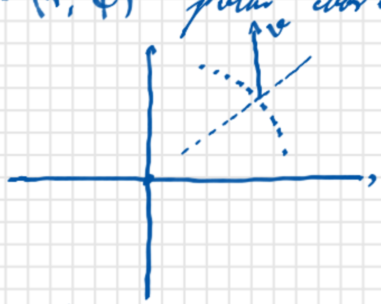
Important example: Each function $f: M \to \mathbb{R}$ defines a 1-form df.

$$df_x(v_x) = v_x(f) = \partial_v f = d_t f(\gamma)$$

For example: $f = \alpha^i$, i-th coordinate function: $v_x^i = d\alpha^i_x(v_x)$

Example: $M = \mathbb{R}^2 \setminus \{0\}$, $\alpha = (r, \phi)$ polar coordinates





cartesian coord.

- $\phi(x) = \tan^{-1}(x^2/x^1)$

  $r(x) = (x^{1\,2} + x^{2\,2})^{1/2}$

$$v_x^\phi = d\phi_x v_x = \partial_v \phi_x = d_t \tan^{-1}\left(\gamma^2(t)/\gamma^1(t)\right)$$

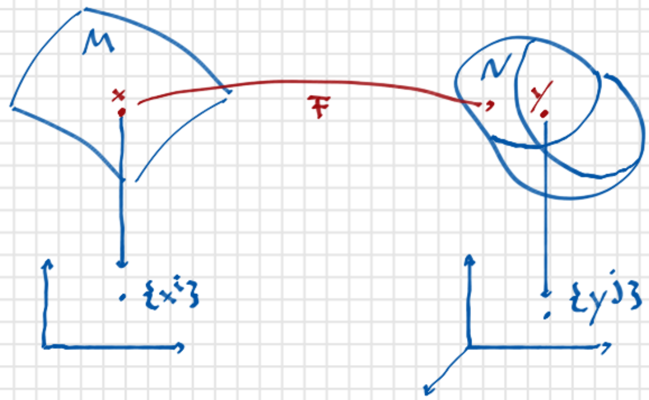
Note: $dx^i = d\alpha^i$ acts on ∂_j as $dx^i \partial_j = \delta^i_j$. $\{dx^i\}$ is basis of TM^*. Every $\phi \in TM^*$ can be expanded as $\phi = \sum_i \phi_i dx^i$.

Summary: With M, $\{(\varphi, U)\}$ and $\varphi^i = x^i$ have:

		TM	TM^*	
elements:	vector fields	v	ϕ	one-forms
basis:	coord. vector fields	$\partial_{x^i} = \partial_i$	dx^i	coordinate forms
decomposition:		$v = v^i \partial_i$	$\phi = \phi_i \, dx^i$	
coefficients		$v^i = v(x^i)$	$\phi_i = \phi(\partial_i)$	

Pushforward & Pullback

general setting:



Realizations: • M = coordinate domain, N = field manifold, F = field

• M = coordinate domain, N = another coord. domain, F = coordinate change

• M = interval, N = manifold, F = curve

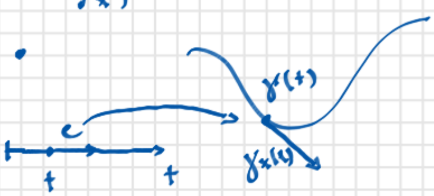
⋮

• A vector field on M can be pushed forward to one on N: $v \mapsto F_* v$

If $v_x \leftrightarrow \gamma$, $F_*(v_x)$ is defined through curve $F(\gamma)$. In components:

$(F_* v)^i = \frac{\partial F^i}{\partial x^j} v^j$ or $v = v^j \partial_{x^j} \mapsto F_* v = \frac{\partial F^i}{\partial x^j} v^j \partial_{y^i}$

Example: •



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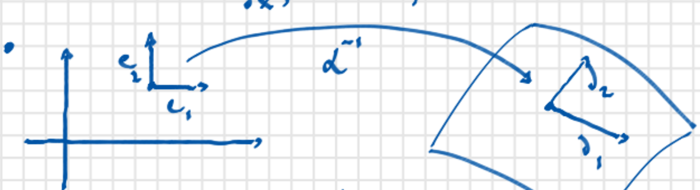


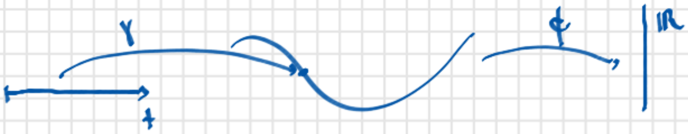
$\partial_i = \varphi^{-1}_* e_i$

- A differential form on N can be pulled back to one on M.

$\omega \mapsto F^*\omega$. For $v \in TM$ $(F^*\omega)(v) = \omega(F_* v)$. In components:

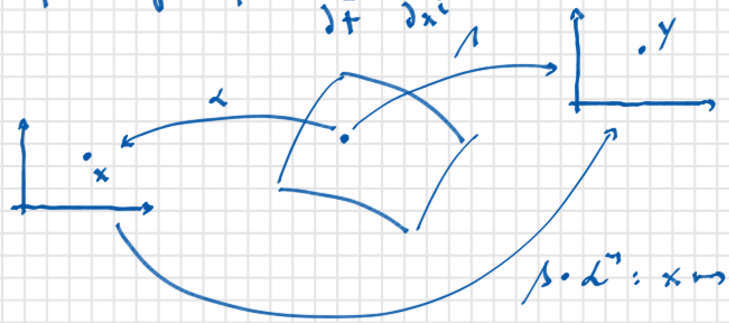
$(F^*\omega)_i = \frac{\partial F^j}{\partial x^i}\,\omega_j$ or $\omega = \omega_j\, dy^j$ $\quad (F_*\omega) = \frac{\partial F^j}{\partial x^i}\,\omega_j\, dx^i$

Example: •



$$d\phi \mapsto \gamma_* d\phi = \frac{\partial \gamma^i}{\partial t}\frac{\partial \phi}{\partial x^i}\, dt$$

•



$\beta \circ \alpha^{-1} : x \mapsto y$

$$dy^j \to \frac{\partial y^j}{\partial x^i}\, dx^i$$

Differential forms of higher degree

Alternating n-forms: Def:

$$\Lambda^p M = \{\omega : TM^* \times \dots \times TM^* \to \mathcal{E}^\infty(M) \mid \omega(v_1 .. v_i .. v_j .. v_p) = -\omega(v_1 .. v_j .. v_i .. v_p)$$
$$\& \ \omega \text{ multi-linear}\}$$

$(\Lambda^p M)_x$ is a vector space with Dimension $\binom{n}{p}$. $\Lambda^0 M \cong \mathcal{E}^\infty(M)$, $\Lambda^{p>n} M = \emptyset$

Def.: Wedge product

$$\wedge : \Lambda^p M \times \Lambda^q M \to \Lambda^{p+q} M$$

$$\omega \times \eta \mapsto \omega \wedge \eta$$

$$(\omega \wedge \eta)(v_1 \dots v_{p+q}) = \frac{1}{p!\,q!} \overbrace{\sum_{S \in S^{p+q}}}^{\text{sum over permutations}} \operatorname{sgn} S\; \omega(v_{S_1} .. v_{S_p})\, \eta(v_{S_{p+1}} .. v_{S_{p+q}})$$

Note: • $\Lambda M = \bigoplus_p \Lambda^p M$ is algebra with product '$\wedge$', the Grassmann algebra

• Every p-form can be represented as $\omega = \frac{1}{p!} \sum_{\{i\}} \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

where $\omega_{i_1 \ldots i_p} \in \mathcal{E}^\infty(M)$ are antisymmetric in indices.

Example: $j = \frac{1}{2} j_{ik}\, dx^i \wedge dx^k$ acts on $v = v^i \partial_i$ and $w = w^i \partial_i$

as $j(v, w) = \frac{1}{2} j_{ik} (v^i w^k - v^k w^i)$

Def.: Exterior derivative: $d : \Lambda^p M \to \Lambda^{p+1} M$, $\omega \mapsto d\omega$

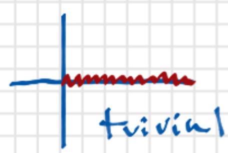
$$\omega = \frac{1}{p!}\, \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \to \frac{1}{p!} (\partial_{x^j} \omega_{i_1 \ldots i_p})\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

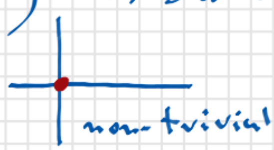
• Example: $\phi \in \Lambda^0 M = \mathcal{E}^\infty M$ $\quad d\phi = \partial_i \phi\, dx^i$

• $d^2 = \emptyset$ $\left(d^2 \omega = (\underbrace{\partial^2_{x^i x^j}}_{\text{sym.}} \omega_{\ldots}) \underbrace{dx^i \wedge dx^j}_{\text{a-sym.}} \ldots \right)$

• $\omega = d\alpha$ ⇝ ω is exact. $d\omega = 0$ ⇝ ω is closed. Every exact form is closed: $\omega = d\alpha \Rightarrow d\omega = 0$. But the reverse is not true.
For example: On $\mathbb{R}^2 \setminus \{0\}$ define: $\omega = \frac{x^1 dx^2 - x^2 dx^1}{x^{1 2} + x^{2 2}}$ (*) $\to d\omega = 0$

But $\nexists\, \phi \in \mathcal{E}^\infty(\mathbb{R}^2 \setminus \{0\})$ $\omega = d\phi$. $\left(\text{Locally: } \omega = d \tan^{-1}\left(\frac{x^2}{x^1}\right) \right)$

• The classification of closed forms which are not exact has to do with the topology of M. For example, on 

, ω (*) is exact, but on 

it is not.

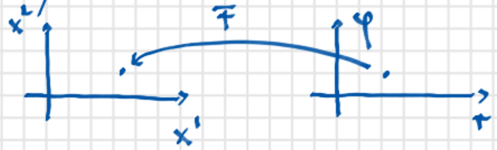
• Pullback of forms of higher degree defined as

$(F^*\omega)(v_1,\dots,v_p) = \omega(F_* v_1,\dots,F_* v_p)$ (Need this later for integration)

In coordinates

$$(F^*\omega) = \frac{\partial F^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial F^{j_p}}{\partial x^{i_p}}\,\omega_{j_1\dots j_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

Example:

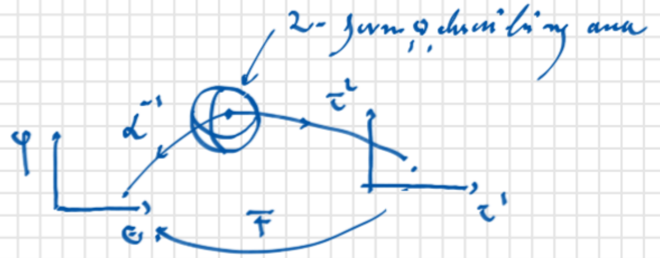
• 

$F^1(r,\varphi) = r\cos\varphi$

$F^2(r,\varphi) = r\sin\varphi$

$\omega = dx^1\wedge dx^2$ $\qquad F^*\omega = \left(\frac{\partial F^1}{\partial r}\frac{\partial F^2}{\partial \varphi} - \frac{\partial F^2}{\partial r}\frac{\partial F^1}{\partial \varphi}\right) dr\wedge d\varphi = r\, dr\wedge d\varphi$

Area form in polar coordinates

• 

. $\alpha^{-1*}\omega = \sin\theta\, d\theta\wedge d\varphi$

. $F^1(\tau) = \theta(\tau)$

$F^2(\tau) = \varphi(\tau)$

$$F^*(\alpha^{-1*}\omega) = \left(\frac{\partial\theta}{\partial\tau^1}\frac{\partial\varphi}{\partial\tau^2} - \frac{\partial\theta}{\partial\tau^2}\frac{\partial\varphi}{\partial\tau^1}\right)\sin\theta\; d\tau^1\wedge d\tau^2$$

$$" = \left(\frac{\partial n}{\partial\tau^1}\times\frac{\partial n}{\partial\tau^2}\right)\cdot n \quad " \quad n = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Area form in generic coordinates.

• For $n = 3$: $\quad \omega = \omega_i\, dx^i$ $\qquad$ coeffs: $(\omega_1, \omega_2, \omega_3) \leftrightarrow \underline{\omega}$

$\alpha = \frac{1}{2}\alpha_{ij}\, dx^i\wedge dx^j$ $\qquad (\alpha_{23}, \alpha_{31}, \alpha_{12}) \leftrightarrow \underline{\alpha}$

$\phi\in C^\infty(M)$: $\quad d\phi = \partial_i\phi\, dx^i \leftrightarrow$ '$\nabla\phi$'

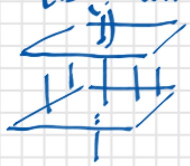
$d\omega = \partial_j\omega^i\, dx^j\wedge dx^i \leftrightarrow$ '$\nabla\times\underline{\omega}$'

$d\alpha = \frac{1}{2}\partial_k\alpha_{ij}\, dx^k\wedge dx^i\wedge dx^j \leftrightarrow$ '$\nabla\cdot\underline{\alpha}$'

• Example **Differential forms of electrodynamics** (non-covariant)

On $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ define

$E = E_i \, dx^i$ $\qquad$ $D = \frac{1}{2} D_{ij} \, dx^i \wedge dx^j$ $\qquad$ $\rho = \rho \, dx^1 \wedge dx^2 \wedge dx^3$



$j = \frac{1}{2} j_{ij} \, dx^i \wedge dx^j$

$H = H_i \, dx^i$ $\qquad$ $B = \frac{1}{2} B_{ij} \, dx^i \wedge dx^j$

$$dD = \frac{4\pi}{c} \rho \qquad dH - \frac{1}{c} \partial_t D = \frac{4\pi}{c} j$$

$$dB = 0 \qquad dE + \frac{1}{c} \partial_t B = 0$$

Material equations: $E \mapsto D$, $B \mapsto H$ require metric (Hodge star). In vacuum $D_{12} = E_3 \dots$, $B_{12} = H_3 \dots$

Covariant formulation: Def.: **field strength tensor**: $F = E \wedge dx^0 + B$ $\quad x^0 = ct$

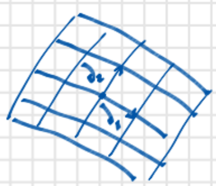
$\rightsquigarrow dF = 0$. $\quad G = \tilde{F} = *F = -H \wedge dx^0 + D \rightsquigarrow dG = 4\pi j$. $\quad dF = 0 \rightsquigarrow F = dA$

$A = A_\mu \, dx^\mu$ vector potential.

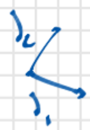
Integration of forms

Picture: • Take d-form ω + d-Manifold M

• Triangulate M



• Feed



into ω and sum

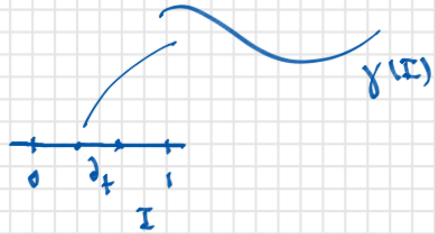
Formal implementation: On chart (α, U):

$$\int_U \omega \equiv \int_{\alpha(U)} (\alpha^{-1})^* \omega = \int_{\alpha(U)} \omega(\underbrace{\alpha^{-1}_* e_1}_{\partial_1}, \dots, \alpha^{-1}_* e_d) \, dx^1 \wedge \dots dx^d \equiv \int_{\alpha(U)} \omega(\partial_1, \dots, \partial_d) \, dx^1 \dots dx^d$$

Riemann int.

Examples :

• $\omega \equiv F$ a 1-form. $I =]0,1[\subset \mathbb{R}$. $\gamma : I \to \mathbb{R}^2$.



$$\int_{\gamma(I)} F = \int_I \gamma^* F = \int_I F(\gamma_* \partial_t)\, dt$$

$$= \int_I F\left(\frac{\partial \gamma^i}{\partial t} \partial_i\right) dt = \int_I \frac{\partial \gamma^i}{\partial t} F_i \, dt$$

• $\omega = j$ a 2-form. $\alpha(U) =]0,1[\times]0,1[\ni (\tau^1, \tau^2) \equiv \tau$. $\alpha^{-1}(\tau) \equiv x(\tau)$

$$\int_U j = \int_{\alpha(U)} x_* j = \int_{]0,1[\times]0,1[} j_{\ell n} \left(\frac{\partial x^\ell}{\partial \tau^1}\frac{\partial x^n}{\partial \tau^2} - \frac{\partial x^\ell}{\partial \tau^2}\frac{\partial x^n}{\partial \tau^1}\right) d\tau^1 d\tau^2$$

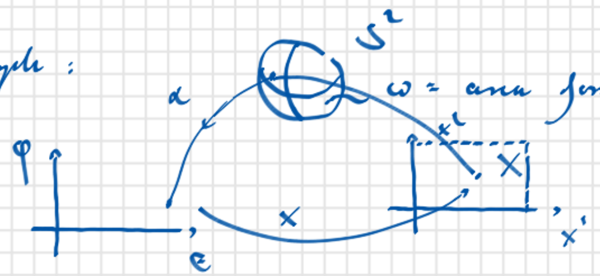
$$\left\{ \begin{array}{l} \cdot\ j = j_{\ell m}\, dx^\ell \wedge dx^m \\ \cdot\ x^* j = j_{\ell n} \dfrac{\partial x^\ell}{\partial \tau^i} \dfrac{\partial x^n}{\partial \tau^j}\, d\tau^i \wedge d\tau^j \end{array} \right.$$

Important property of definition: $F : U \to F(U) \equiv V$, $\dim V = \dim U = n$
diffeom.

$\omega \in \Lambda^n(V)$

$$\int_{F(U)} \omega = \int_U F^* \omega$$

Example :



$$\alpha^{-1*}\omega = \sin\theta \, d\theta \wedge d\varphi \quad \rightsquigarrow \quad \int_{S^2} \omega = \int \sin\theta \, d\theta \, d\varphi = 4\pi$$

$$= \int_X x^*(\alpha^{-1*}\omega) = \int_X \sin\theta(x) \left(\frac{\partial \theta}{\partial x^1}\frac{\partial \varphi}{\partial x^2} - \frac{\partial \theta}{\partial x^2}\frac{\partial \varphi}{\partial x^1}\right) dx^1 \wedge dx^2$$

$$= \int_X \varepsilon_{ijk}\, n^i \frac{\partial n^j}{\partial x^1} \frac{\partial n^k}{\partial x^2}\, dx^1 \wedge dx^2$$

$$n = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)^T$$

Stokes Theorem (not rigorous). M a d-dimensional manifold with $(d-1)$-dimensional boundary ∂M. $\phi \in \Lambda^{d-1}(\partial M)$.

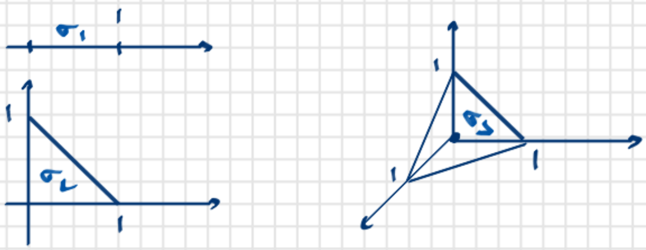
$$\int_{\partial M} \phi = \int_M d\phi$$

Example: $\partial D^1 = S^1 \subset \mathbb{R}^2$ $\quad \phi = \frac{1}{2} r^2 d\varphi$ $\quad$ |

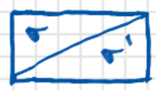
$$\frac{2\pi}{2} = \int_{S^1} \phi = \int_{D^1} r\,dr\,d\varphi = \pi$$

Chains and simplices:

A **d-simplex**: the simplest polyhedron in d-dimensional space.



d-standard simplices.

- Simplices can be linearly combined to chains.  $\tau = \sigma + \sigma'$
- A **d chain** is the formal linear combination of d-simplices with integer coefficients: $\sigma = m_1\sigma_1 + m_2\sigma_2 + \dots$
- The **boundary** $\partial\sigma$ of a d simplex σ is the alternating (!) sum of its bounding faces.

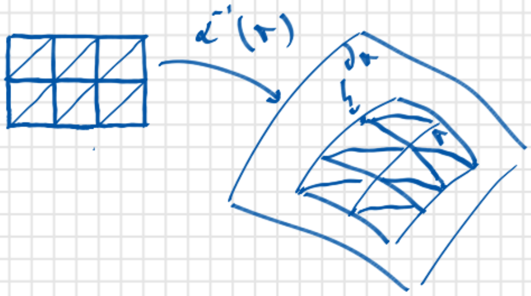
$$\partial(\Delta) = \text{—} - \backslash + |$$

$$\partial\,(\times\!\!-\!\!\bullet) = \bullet - \times$$

The definition entails that $\partial \cdot \partial = 0$!

- Define: C_p = space of p-chains (It's a space if we allow for real coefficients. $r\sigma + r'\sigma'$.) $\partial: C_p \to C_{p-1}$, $\sigma \mapsto \partial\sigma$ is a linear map.

Chains can be embedded into Manifolds to 'triangulate' them



Stokes in chain language:

$$\int_{\partial\sigma}\omega = \int_{\sigma} d\omega$$

(Do not discriminate between $\sigma = \varphi^{-1}(\sigma)$)

Homology : On M, consider:

$C_p(M) \ni p$ - chains (an abelian group)

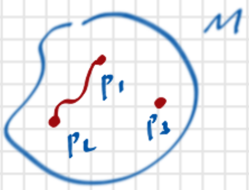
$Z_p(M) = \{ \underbrace{z_p}_{p\text{-cycles}} \in C_p(M) / \partial z_p = \emptyset \}$ p-cycle group

$B_p(M) = \{ z_p \in C_p(M) / \ \exists\, w_{p+1} \in C_{p+1}(M) ,\ z_p = \partial w_{p+1} \}$ p-boundary group

$B_p(M) \subset Z_p(M) \rightsquigarrow$ define $Z_p(M) / B_p(M) = H_p(M)$ p-homology group

$H_p(M)$ describes the topology of M.

0-th homology group and connectivity:

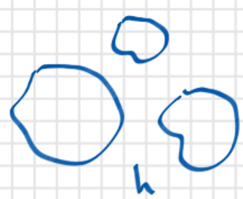


(Boundaryless) 0-cycles: $\sigma = m_1 p_1 + m_2 p_2 + m_3 p_3 \cdots$

Consider $\sigma' = m_1 (\underset{p_1 \ \ p_2}{\bullet\!\sim\!\bullet})$. $\partial\sigma' = m_1 (\underset{p_1}{\bullet} - \underset{p_2}{\bullet})$ a 0 boundary.

$$\sigma \sim \sigma - \partial\sigma' = (m_2 - m_1) p_2 + m_3 p_3$$

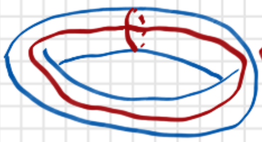
Iterate: $\sigma \sim m p_1 \rightsquigarrow H_0(M) \simeq \mathbb{Z}$. On M'



$H_0(M') \simeq \mathbb{Z}^k$

1 - Homology and topology. Example: T^2 the 2-torus.

$\sigma_2(T) \equiv$ Triangularization of T^2. $\partial\sigma_2(T) = 0$. No other boundary-less 2d-chain on T^2 ⇝ $H_2(T^2) = \mathbb{Z}$



$\partial c_i = 0 \quad c_i \neq \partial\sigma_2 \quad ⇝ \quad H_1(T_2) = \mathbb{Z}^2$

$H_0(T_2) = \mathbb{Z}$

$H_\ell(M)$ $\ell \in d$ ⇝ classifies no. of $d-\ell$ dimensional 'holes' in M

Both homotopy and homology groups describe holes in manifolds. But: homology 1) easier to calculate, and 2) establishes connection to differential forms via co-homology (this is important in field theory.)

Cohomology: On $\Lambda^p M$ define

- $Z^p(M) = \{\omega \in \Lambda M \mid d\omega = 0\}$ p-cocycle group (under addition of forms)
- $B^p(M) = \{\omega \in \Lambda M \mid \omega = d\alpha\}$ p-coboundary group
- $H^p(M) = Z^p(M)/B^p(M)$ p-cohomology group

Cohomology and homology group are isomorphic to each other. For example: $M = \mathbb{R}^2 \setminus \{0\}$: $H_1(M)$ generated by circle around origin $H_1(M) = \mathbb{Z}$ or $\mathbb{R}$ if we allow for integer coefficients. $H^1(M)$ generated by angle form ω, i.e. form with local (but not global) representation $d\varphi$.

Isomorphy is established by Stokes theorem:

$$\langle\, , \,\rangle : C_p(M) \times \Lambda^p(M) \to \mathbb{R}$$
$$(\sigma \;,\; \omega) \mapsto \langle \sigma, \omega \rangle = \int_\sigma \omega$$

a scalar product. Interpret chains as elements of $(\Lambda^p(M))^*$.

the boundary operator is self adjoint relative to $\langle\,,\rangle$: $\langle \partial\sigma, \omega\rangle = \langle \sigma, d\omega\rangle$ by Stokes. $\langle\,,\rangle$ establishes isomorphy between homology groups. E.g. $\tau \in B_p(M)$ a boundary, and $\omega \in Z^p(M)$ a cocycle $\leadsto$ $\tau = \partial\tilde{\sigma}$, $d\omega = 0$ $\leadsto$ $\langle \tau, \omega\rangle = \langle \partial\tilde{\sigma}, \omega\rangle = \langle \tilde{\sigma}, d\omega\rangle = 0$. For any $\sigma \in Z_p(M)$ $\leadsto$ $\langle \sigma + \tau, \omega\rangle = \langle \sigma, \omega\rangle$ $\leadsto$ $\langle\,,\rangle$ depends only on $Z_p(M)/B_p(M) = H_p(M)$.

Take home message: For any nontrivial 'hole' of dimension d on a Mf there exists one closed d-form that describes it.

Motivating the field theory of the IQH

- Generating functional for Green functions:

$$Z = \int \mathcal{D}(\bar\psi,\psi)\ \exp\left(i\,\bar\psi^a_s\left(E_F + (-)^s(\omega + i0) - \hat H\right)\psi^a_s\right)$$

$\psi^a_s(x)$: Grassmann field. $a = 1,\dots,R$ replica index, $s = \pm 1$

From Z can compute $G^\pm(\pm\omega, x, x') = \langle x|(E_F \pm \omega \pm i0 - \hat H)^{-1}|x'\rangle$

- Action symmetric under $\psi \to T\psi$ $\quad T \in U(2R)$ (at $\omega = 0$)
- Disorder average: $i0 \to \frac{i}{2\tau}$ (scattering rate)
- Symmetry broken to $U(R)\times U(R)$ $\sim$ Goldstone mode manifold $U(2R)/U(R)\times U(R)$ parametrized by $Q = T\tau_3 T^{-1}$

- Goldstone mode action in 2d and presence of magnetic field. Criteria:
  - symmetry explicitly broken by ω
  - Rotational invariance
  - Non-invariant under $x\to -x$ or $y \to -y$.

$$S[Q] = \frac{1}{8}\int d^2x\left(\sigma_{xx}\,\mathrm{tr}\left(\partial_\mu Q\,\partial_\mu Q\right) - 4\pi\nu i\omega\,\mathrm{tr}\left(Q\tau_3\right) - \sigma_{xy}\,\epsilon_{\mu\nu}\,\mathrm{tr}\left(Q\partial_\mu Q\partial_\nu Q\right)\right)$$

For $\omega = 0$, action governed by 2 parameters, σ_{xx}, σ_{xy}. Term $\propto \sigma_{xy}$ is Θ-term (see below)

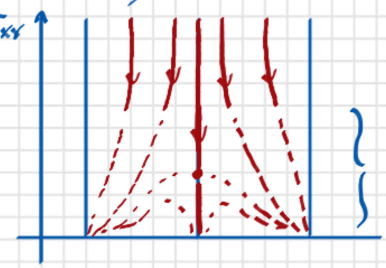
- Field theory must be analyzed by RG methods. For $\omega = 0$, $\sigma_{xy} = 0$: $\frac{d\sigma_{xx}}{d\ln L} < 0$ one-parameter scaling.

• RG for $\sigma_{xy} \neq 0$: Theta term perturbatively invisible (topology does not change under small fluctuations). RG in the presence of instantons (schematically)

$$\frac{d\sigma_{xx}}{d\ln L} = -\frac{1}{2\pi^2\sigma_{xx}} - c\,\sigma_{xx}\, e^{-2\pi\sigma_{xx}} \cos(2\pi\sigma_{xy})$$

$$\frac{d\sigma_{xy}}{d\ln L} = c\cdot\sigma_{xx}\cdot \underbrace{e^{-\pi\sigma_{xx}}}_{\text{the action cost of instantons.}} \sin(2\pi\sigma_{xy})$$

⇝ 2 parameter scaling



Topology of Θ-term & boundary theory.

$\phi: S^2 \to T = U(2R)/U(R)\times U(R)$, $\pi_2(T) = \mathbb{Z}$

$x \mapsto G(x)$, S^2 = compactified space.

On T consider 2-form: $\omega = \mathrm{tr}(G\, dG \wedge dG)$

$S_{top}[G] = -\frac{\sigma_{xy}}{8}\int_{S^2} \phi^*\omega$. $\phi^*\omega$ is 2-form on 2 dim manifold ⇝ locally exact.

May verify that $\omega = d\kappa$ $\quad \kappa = 4\,\mathrm{tr}(T\tau_3\, dT^{-1})$ ← coordinate representation (local)

Consider theory with boundary $S^2 \to D$, a disk. Assume D to be asymptotically large, so that $(\sigma_{xx}, \sigma_{xy}) \xrightarrow[L\to\infty]{} (0, n)$.

$$⇝ S[G] = -\frac{n}{8}\int_D \phi^*\omega = -\frac{n}{8}\int_D \phi^* d\kappa = -\frac{n}{8}\int_D d(\phi^*\kappa) = -\frac{n}{8}\int_{\partial D}\phi^*\kappa$$

$$= -\frac{n}{2}\int ds\, \mathrm{tr}(T\tau_3\, \partial_s T^{-1})$$

Theory reduces to boundary theory. Consistency requires $n \in \mathbb{Z}$: $Q = T\tau_3 T^{-1}$ invariant under $T \to Th$, $[h, \tau_3] = 0$: $h = \mathrm{diag}(h_+, h_-)$ $h_\pm \in U(R)$. $\det(h) = 1$

But: $S_{top}[T] \to \frac{n}{2} \sum_{\sigma=\pm} \int \mathrm{tr}(h_\sigma \partial_s h_\sigma^{-1})\, \sigma\, ds = -\frac{n}{2} \sum_\sigma \int \partial_s \mathrm{tr} \ln h_\sigma\, \sigma =$

$$= -\frac{n}{2} \sum_\sigma \sigma \ln \det h_\sigma = -i\frac{n}{2} \sum_\sigma \underbrace{(\phi_\sigma(L) - \phi_\sigma(0))}_{2\pi W_\sigma} \sigma = -i\pi n \sum_\sigma W_\sigma \sigma =$$

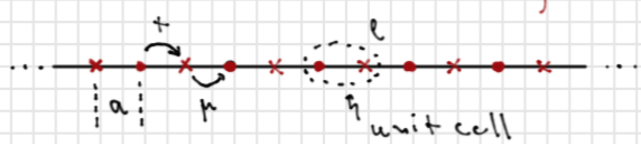
$$\det h_\sigma = e^{i\phi_\sigma} \tilde{h}_\sigma \qquad \tilde{h}_\sigma \in SU(R)$$

$$\underset{\det h = 1}{=} -2\pi i n W_+ \quad \text{unobservable for } n \in \mathbb{Z}$$

Conclusion: Localization $((\sigma_{xx}, \sigma_{xy}) \to (0, \text{integer}))$ and existence of well defined boundary theory depend on each other.

A simpler example of similar physics: 1d topological insulator

The Su-Schrieffer-Heeger (SSH) chain (1979)

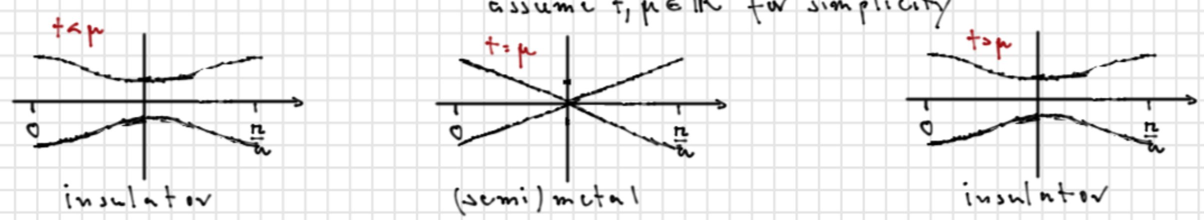


(periodic b.c.)

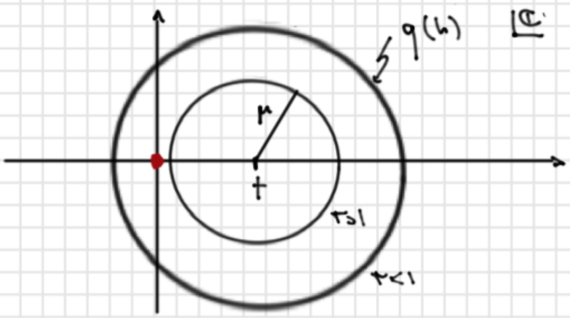
Diagonalize problem in terms of 2-component Bloch wave functions

- $\psi_k(l) = \begin{pmatrix} \psi_\times \\ \psi_\bullet \end{pmatrix}_k e^{ikl}$ $\quad k = 0, \frac{2\pi}{L}, \dots, \frac{\pi}{a}$ $\quad L = N(2a) =$ chain length
- $\hat{H}_k = \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix} \begin{matrix} \times \\ \bullet \end{matrix}$ $\quad q_k = t - e^{ik\cdot 2a} \mu$
- Eigenvalues: $\epsilon_k^\pm = \pm |q_k| = \left(t^2 + \mu^2 - 2t\mu \cos(2ka)\right)^{1/2}$

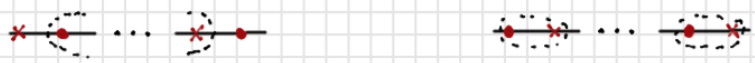
assume $t, \mu \in \mathbb{R}$ for simplicity



- interpretation: $r = \frac{t}{\mu} = 1$ marks quantum phase transition between two distinct insulating phases. No change in symmetry, no local order parameter.
- topological order: ground states of $r<1$ and $r>1$ carry distinct topological invariant



- curve $S^1 \to \mathbb{C}\setminus\{0\}$
  $k \mapsto q(k)$
  homotopically trivial/non-trivial for $r>1$ / $r<1$
- observable difference: cut system open depending on

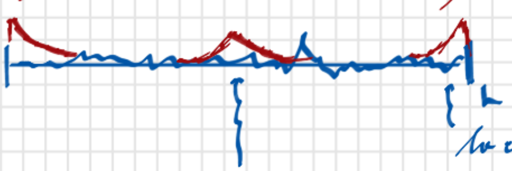


the system with $r>1$ has/has not two zero energy boundary states sharply ($\mathcal{O}(a)$) localized at system boundaries. The insulating $r>1$ phase has a 'conducting surface'.

- Summary: $\exists$ (second order) topological quantum phase transitions without symmetry changes/local order parameter.

Field theory of the disordered topological chain.

Qualitative view:



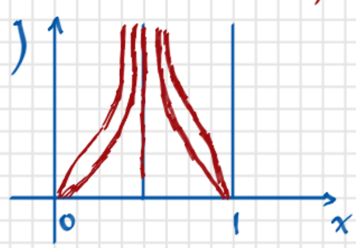
off criticality

localized 0-energy boundary states in top. phase

localized finite energy generic states.

Topological phase transition $\mu = t$: localization length diverges $\xi \to \infty$

System characterized by 2 parameters: 1) $\xi \leftrightarrow g$ = longitudinal conductance 2) $\chi = \langle n \rangle$ disorder average of topological index. Expect flow:



I.e. for $L \to \infty$ $g \to 0$ localization and $\chi \in \{0,1\}$ self-averaging of top. index.

Topological field theory:

Action: $S = \bar\psi^a_s \left(\omega + s(\omega + i0) - \hat H \right) \psi^a_s$ $\qquad \psi_s = \begin{pmatrix} \psi_\times \\ \psi_\circ \end{pmatrix}_s \qquad \hat H = \begin{pmatrix} & Q \\ Q^+ & \end{pmatrix}$

invariant under: $\bar\psi_\circ \to \bar\psi_\circ U_L^{-1} \qquad \psi_\circ \to U_R \psi_\circ$ $\qquad$ for $\omega = 0$.

$\qquad \bar\psi_\times \to \bar\psi_\times U_R^{-1} \qquad \psi_\times \to U_L \psi_\times$

Invariance under $U(2R)\times U(2R)$. Note: at low energies and no disorder:
$q_k = (t-p) - 2ipak \rightsquigarrow H = (t-p)\sigma_1 + 2pa(i\partial)\sigma_2 \rightsquigarrow$ Dirac Hamiltonian $\rightsquigarrow$ chiral sym. of 1d Dirac theory.
Disorder averaging: $i\delta \to \frac{i}{2\tau}$ symmetry broken to $U_L = U_R$. Goldstone mode mf.
$U(2R)\times U(2R) / U(2R) \simeq U(2R)$.

Such field theory characterized by following criteria:

- field: $T : [0,L] \to U(2R)$, $x \mapsto T(x)$
- For $\omega = 0$ action invariant under $T \to U_L T U_R$
- For $p = t$, action invariant under $x \to -x$.

$$S[T] = \int_0^L dx \left(+ \frac{\xi}{4} \mathrm{tr}(\partial_x T \partial_x T^{-1}) + \chi\, \mathrm{tr}(T^{-1}\partial_x T) + \kappa\nu\omega\, \mathrm{tr}((T+T^{-1})\sigma_3) \right); \quad \chi \propto t-p+\tfrac{1}{2}$$

Can successively integrate out T (at $\omega = 0$) to generate flow $(\xi(L), \chi(L))$. Confirms conjecture above. For large L and $\chi \neq \frac{1}{2}$, $\chi \to n$, $\xi \to 0$. Action collapses to

$$S_{top}[T] = n \int_0^L dx\, \mathrm{tr}(T^{-1}\partial_x T) = n \int_0^L dx\, \partial_x \mathrm{tr} \ln T = n \int_0^L dx\, \partial_x \ln \underbrace{\det T}_{= e^{i\phi}} =$$

$$= in\left(\ln\phi(L) - \ln\phi(0)\right) \rightsquigarrow \ldots \rightsquigarrow \text{ for } n = 1 : \text{ boundary state.}$$

Bosonization reminder.

Action of 1d chiral fermions: $S[\bar\chi,\chi] = \int d^2x\; \bar\chi(\partial_0\gamma_0 - i\partial_3\gamma_3)\chi$

$\chi = \begin{pmatrix}\chi_L \\ \chi_R\end{pmatrix} \qquad x_0 = c\tau$

- Action has $U(1)\times U(1)$ invariance under $\chi_c \to e^{i\phi_c}\chi_c,\ \bar\chi_c \to \bar\chi_c e^{-i\phi_c}$, $c = L, R$
- Correlation functions, e.g. $\langle(\bar\chi_L\chi_R)(x)(\bar\chi_R\chi_L)(0)\rangle = \frac{1}{(2\pi)^2}\frac{1}{x^2}$
- Theory equivalent to bosonic one.

$$S[\theta,\phi] = \frac{1}{2\pi}\int dx\,d\tau\left(2i\partial_\tau\theta\,\partial_x\phi + (\partial_x\theta)^2 + (\partial_\tau\phi)^2\right)$$

Hamiltonian action of field θ with canonical momentum $\partial_x\phi/\pi$.

Lagrangian representation:

$$S[\theta] = \frac{1}{2\pi}\int dx\,d\tau\left((\partial_x\theta)^2 + (\partial_\tau\theta)^2\right)$$

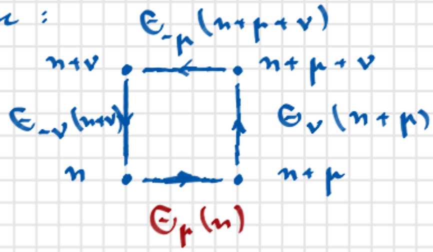
- With identification $\chi_{L/R} = C\,e^{i(\phi\pm\theta)}$ both theories are fully equivalent.
- $U(1)\times U(1)$ symmetry realised as $\phi \to \phi + (\phi_L+\phi_R)/2$
  $\theta \to \theta + (\phi_L - \phi_R)/2$
- Fermion density: $\rho_{L/R} = \frac{1}{2\pi}(\partial_x\theta \mp \partial_x\phi) \rightsquigarrow$ Fermion interactions $\rho_L\rho_R$

(complicated) $\to (\partial_x\theta)^2 + (\partial_x\phi)^2$ quadratic and easy! However $\bar\chi_L\chi_R$ (easy) $\to e^{-2i\theta}$ (complicated)

Lattice gauge theory in a nutshell (Review: Kogut, 79)

Example: $U(1)$ lattice gauge theory on a 4d lattice – discrete Electrodynamics.

Consider lattice structure:



- $\Theta_\mu(n) \in [0, 2\pi]$
- $\Theta_{-\mu}(n+\mu) = -\Theta_\mu(n)$

Goal: Define theory of $\{\Theta_\mu(n)\}$ that contains a local gauge symmetry defined at the sites, n, of the lattice.

Def.: $$S[\Theta] = \frac{1}{2g^2} \sum_{\text{plaquettes}} \prod_{\circlearrowleft \text{ links}} e^{i\Theta_\mu(n)}$$

Action is invariant under local transformation $\Theta_\mu(n) \to \Theta_\mu(n) + \chi(n) - \chi(n+\mu)$ where $\chi(n) \in [0, 2\pi]$ is a lattice function

Relation to continuum electrodynamics: Define a: lattice spacing, and $\Theta_\mu(n) \equiv a\, g\, A_\mu(n)$. For small g and smooth A:

$$\sum \prod_{\circlearrowleft} e^{i\Theta_\mu(n)} = \sum \prod_{\circlearrowleft} e^{i a g A_\mu(n)} = \sum e^{i a g \sum_{\circlearrowleft} A_\mu(n)} \simeq$$

$$\simeq i a g \sum \sum_{\circlearrowleft} A_\mu(n) - \frac{1}{2} a^2 g^2 \sum \Big(\sum_{\circlearrowleft} A_\mu(n) \Big)^2$$

$$\left[\begin{aligned} \sum_{\circlearrowleft} A_\mu(n) &= A_\mu(n) + A_\nu(n + a e_\mu) - A_\mu(n + a e_\nu) - A_\nu(n) \simeq \\ &\simeq -a \partial_\nu A_\mu(n) + a \partial_\mu A_\nu(n) \end{aligned} \right.$$

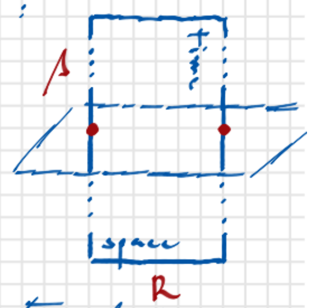
$$\simeq -\frac{1}{4} \int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

$\rightsquigarrow$ the (euclidean) action of electrodynamics. Gauge transformation: $A_\mu \to A_\mu + \partial_\mu \chi$

How does the discrete theory differ from the continuum limit?

Elitzur theorem: gauge non-invariant observables (such as $e^{i\epsilon_\mu(n)}$) have vanishing expectation values.

Good gauge invariant observables: $e^{-i\sum_{\mathcal{C}}\epsilon_\mu(n)}$ when the sum is over closed loop (Wegner-Wilson loops). An interesting reference loop:

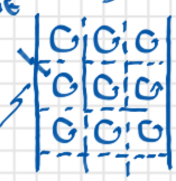


Continuum formulation of loop: $L = \exp\left(-i\oint A_\mu dx^\mu\right)$. Closed unit current loop. At '•': $j_0 = \pm 1$, i.e. loop describes two unit charges separated by distance R.

Strong coupling: $g \gg 1$, large fluctuations of phase variables. Need to fill area of loop with plaquette operators to effect phase cancellation $e^{-i\epsilon}$

$$\langle L\rangle \sim \left(\frac{1}{2g^2}\right)^{R\cdot\beta} \sim e^{2\ln g\, R\cdot\beta} \qquad \text{'area law'}$$

$e^{i\epsilon}$

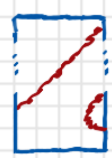


Free energy of charge system grows linearly in distance strong confinement.

Weak coupling: Work in continuum representation: $\langle A_\mu(x)\, A_\nu(x')\rangle = \delta_{\mu\nu}\frac{1}{2\pi^2}\frac{1}{|x-x'|^2}$ for $|x-x'| > a$ in Feynman gauge ($\Rightarrow \Box A_\mu = 0$ as equations of motion).

$$\langle L\rangle = \left\langle e^{ig\oint A_\mu dx^\mu}\right\rangle = e^{-\frac{1}{2}g^2\oint\oint dx^\mu dx'^\nu \langle A_\mu(x)\, A_\nu(x')\rangle} =$$

$$= e^{-\frac{1}{2}g^2 \oint\oint_{\text{parallel}} dx^\mu dx'^\mu \frac{1}{2\pi^2}\frac{1}{|x-x'|^2}} = \dots \text{ some work } \dots =$$



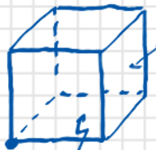
$$\underset{\beta \gg R}{=} e^{-\frac{1}{2}g^2 \underbrace{c}_{\text{const.}} P + \frac{g^2}{2\pi}\frac{\beta}{R} + \frac{2g^2}{\pi^2}\ln(R/a)} \qquad P = 2(T+R) = \text{loop perimeter}$$

Interpret $\beta = T^{-1}$ and $\langle L\rangle = Z(g)/Z(0) \Rightarrow -\beta^{-1}\ln\langle L\rangle = +F(g) - F(0)$

For $\beta \to \infty$: $F(g) - F(0) = \frac{g^2}{2\pi}\frac{1}{R} + \text{const.}$ Deconfinement (+ Coulomb law) if $g = e$ is identified.

Q: Consider Z as imaginary time field integral of lattice QED. What is its Hamiltonian

A: Take continuum limit in t ω direction. Fix gauge $A_0 = 0$, or $\Theta_0(n) = 0$



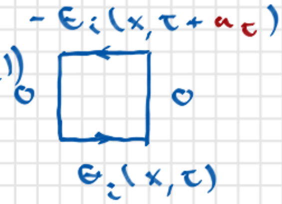
spatio-temporal plaquette, g_τ

$n = (x,\tau)$

$g_\tau \ll g_s$

spatial plaquette, coupling

Spatio-temporal plaquette: $\frac{1}{2g_\tau^2} e^{i(\Theta_i(x,\tau) - \Theta_i(x,\tau+a_\tau))}$



$$\to -\frac{1}{2g_\tau^2} a_\tau^2 \left(\partial_\tau \Theta_i(x,\tau)\right)^2$$

$$S = \frac{a_\tau}{2g_\tau^2} \int d\tau \sum_x \partial_\tau \Theta_i \partial_\tau \Theta_i + \frac{1}{2g_s^2 a_\tau} \int d\tau \sum_x \prod_{\square,\,\text{spatial}} e^{i\Theta_i(x)}$$

Def: $a_\tau g_\tau^{-2} \equiv g^{-2} a \qquad a_\tau^{-1} g_s^{-2} \equiv g^{-2}$

$$= \frac{a}{2g^2} \int d\tau \sum_x \partial_\tau \Theta_i \partial_\tau \Theta_i + \frac{1}{2g^2} \int d\tau \sum_x \prod_{\square} e^{i\Theta_i(x)} \equiv \int d\tau\, L(\Theta,\dot\Theta)$$

Canonical momenta of variables $\Theta_i(x)$: $\pi_i(x) \equiv \frac{\partial L}{\partial \partial_\tau \Theta_i(x)} = a g^{-2} \partial_\tau \Theta_i(x)$

Hamiltonian action:

$$S = \int d\tau \sum_x \left(\pi_i \, \partial_\tau \Theta_i(x) + \frac{1}{2}\left(\frac{g^2}{a}\pi_i^2(x) + \frac{1}{g^2}\prod_{\square} e^{i\Theta_i(x)}\right)\right)$$

$$= \int d\tau \sum_x \left(\pi(x)\, \partial_\tau \Theta_i(x) + \mathcal{H}(\Theta_i, \pi_i)\right)$$

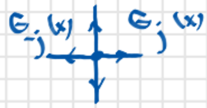
Hamiltonian density

recall $A_i(x) = \frac{1}{ag}\Theta_i(x)$. In gauge $A_0 = 0$: $\partial_\tau A_i(x) = E_i(x)$, electric field.

$$\to \partial_\tau \pi_i(x) = a^2 g^{-1} \partial_\tau A_i(x) = a^2 g^{-1} E_i(x)$$

$\leadsto$ Hamiltonian: $H = \sum_x \mathcal{H} \to \frac{a^3}{2} \sum_x E_i^2(x) + \frac{1}{2g^2} \sum \prod_{\square} e^{i\Theta_i(x)} \underset{g\to 0}{\to} \frac{1}{2}\int d^3x \left(E_i^2 + B_i^2\right)$

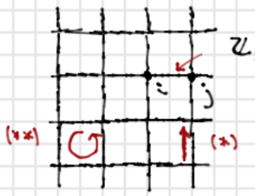
Generators of gauge transformations: $[\Theta_i(x), \pi_j(y)] = -i\,\delta_{ij}\,\delta_{xy}$

$\leadsto \quad e^{i\sum_{\pm j} \pi_j(x)\,\chi}$ shifts all  $\Theta_{\pm j}(x) \to \Theta_{\pm j}(x) + \chi$

$\mathcal{O} = e^{i\sum_{x,j} \pi_j(x)\,\chi(x)}$ generates gauge trafo: $\mathcal{O}\,\Theta_j(x)\,\mathcal{O}^{-1} = \Theta_j(x) - \chi(x) + \chi(x+j)$

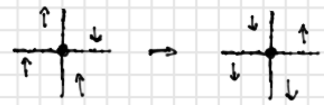
$\mathbb{Z}_2$-lattice gauge theory (Wegner 71, review Kogut RMP 79)

- $\mathbb{Z}_2$- degrees of freedom frequently emerging in correlated fermion systems (Senthil & Fisher, PRB 62, 7850 (2000)): $c^\dagger \to e^{i\phi} \cdot f^\dagger = e^{i\phi} (-1)\times(-1)\, f^\dagger$; fermion charge ; neutral fermion
  $\rightsquigarrow$ an emergent 'gauge degree of freedom'
- Formulate $\mathbb{Z}_2$ gauge theory on a lattice. (Here 2d $\square$-lattice for simplicity)



$\mathbb{Z}_2$- d.o.f. $|\pm\rangle_{ij}$ eigenstates of op $\sigma_{z,ij}$.

- local gauge trafo at i acts through $\prod_{n.n.\ i} \sigma_{x,ij}$
  $\rightsquigarrow |\pm\rangle_{ij} \to |\mp\rangle_{ij}$



- dynamical players of $\mathbb{Z}_2$ lattice gauge theory
  - gauge field along link $i \to j$: $\sigma_{z,ij}$ $\left(\hat{=}\ e^{iA_{ij}}\right.$ in $U(1)$ lattice ED$\left.\right)$
  - electric flux through link $i \to j$: $\sigma_{x,ij}$ (*). $\quad \sigma_z \sigma_x \sigma_z = -\sigma_x \quad \left(\hat{=}\ e^{iA} E e^{-iA} = E+1\right)$
  - magnetic flux through plaquette $\square$: $\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}$ (**). $\left(\hat{=}\ e^{iA_{ij}} \cdots e^{iA_{li}}\right)$

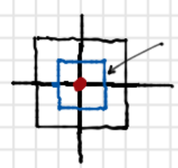
  electric and magnetic flux are gauge invariant

- gauge invariant Hamiltonian

$$\hat{H} = -\gamma \sum_{\text{links}} \overset{(*)}{\sigma_{x,ij}} - \lambda \sum_{\square} \overset{(**)}{\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}}$$

- system supports quantum phase transition driven by $r \equiv \lambda/\gamma$
  1) confining phase $r \leq 1$
  2) topological phase $r \geq 1$

1) Confining phase



flip all spins cut by this dual lattice loop

- Define 'charge density op' : $\hat{\rho}_i = \prod_j \sigma_{x,ij}$
  Charge is locally conserved by $\hat{H}$; $[\hat{H}, \hat{\rho}_i] = 0$
- Ground state in charge neutral sector : $\forall i \quad \hat{\rho}_i |\Psi\rangle = 0$:
  $\sigma_{x,ij} = 1$ globally

- Q: What is ground state in sector of Hilbert space with two charges sitting at i, j?

A: ($\lambda = 0$)

a minimal electric flux line. Costs energy $2\gamma\, d(i,j) = \sigma\, d(i,j)$

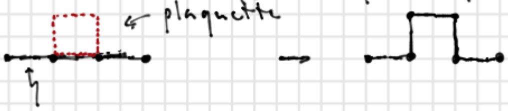
Manhattan

string tension

Energy grows linearly with distance: confinement

- Q: What is the effect of the flux term on this

A:



$\sim$ string fluctuations $\sigma \to 2\gamma - \lambda^2/4\gamma$

strength of pert.

excitation energy

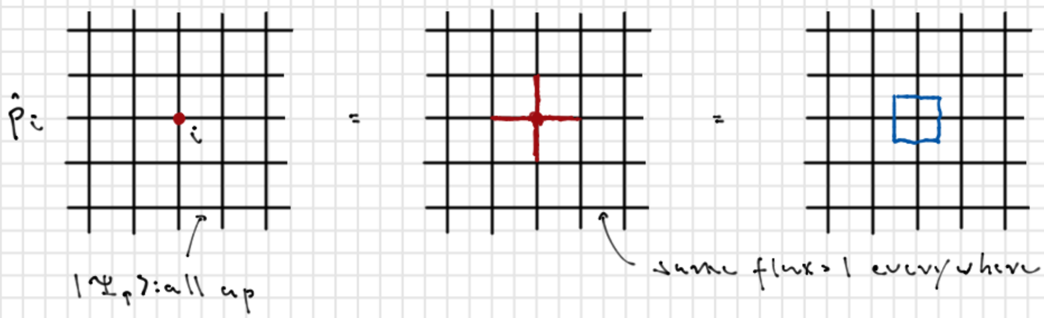
magnetic flux term softens string tension

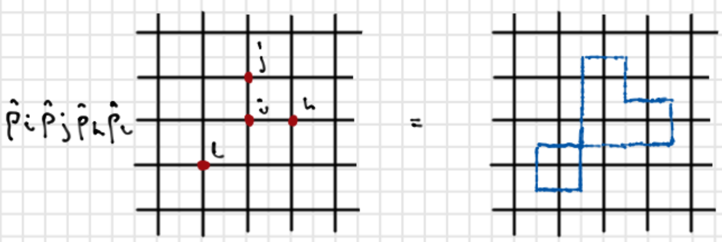
2) Spin liquid phase

- Consider limit $\gamma = 0$. Ground state has flux $\prod_{\square} \sigma_z \ldots \sigma_z$ on each plaquette. This is an implicit characterization.

- Q: How does the ground state in a sector of fixed charge look like?

A: Start from all spins $\sigma_{z,ij} = 1$ state $|\Psi_p\rangle$. This is not a charge eigenstate





ground state: state of all equal weight superposition of closed strings (cf. BCS), a string net condensate

Corrolary: charges totally deconfined.

Q: Is the ground state unique?

A: Def.: V: no. of vertices, E: no. of edges, F: no. of faces

Compactify surface (for simplicity), e.g. . Counting:

E d.o.f. (the spins)

$-(V-1)$ charge constraints (-1 is overall charge neutrality)

$-(F-1)$ flux constraint (-1 is overall flux neutrality)

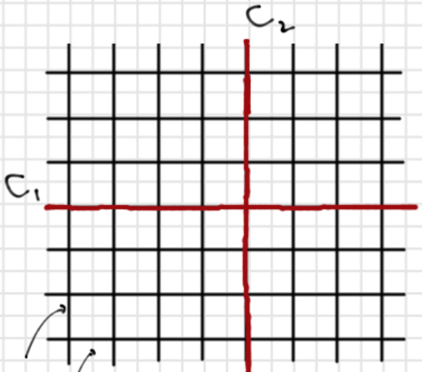
$\underbrace{-F+E-V}_{}\ - 2$ qubit d.o.f. remain

(−) Euler-Characteristic $\chi = 2-2g$. Surface genus g.

$\sim$ ground state degeneracy: 2^{2g}. A hallmark of topological matter.

Q: How do we characterize different ground states?

A:

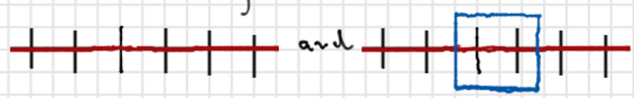


assume periodic boundary conditions a torus, T^2

$$\sigma_z^a = \prod_{C_a} \sigma_{z,ij}.$$

or any other curve winding around T^2.

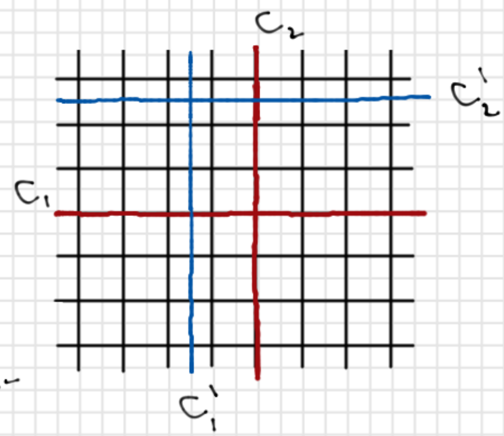
Claim: $\{\sigma_z^1, \sigma_z^2\}$ are topological invariants of each charge sector. E.g.



same C^1

Invariants changed by nonlocal operators.

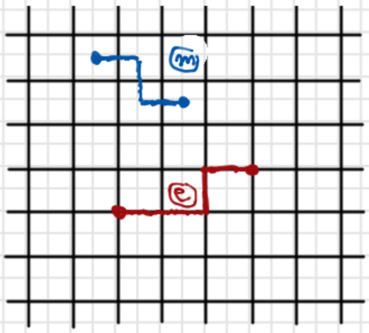
$$\sigma_x^a = \prod_{C'_a} \sigma_{x,ij} \qquad [\sigma_x^a, \sigma_z^a]_+ = 0$$



Global qubits $\{\sigma_x^a, \sigma_z^a\}$ provide topological characterization of ground state $\rightsquigarrow$ topological quantum computation.

Q: What are excitations of the system?

A:



m: a 'magnetic' excitation $= \prod_{\substack{\text{all links} \\ \text{cut by string}}} \sigma_{x,ij}$

changes flux of terminal plaquettes. Costs energy 2λ.

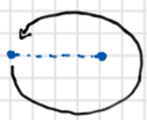
e: an 'electric excitation' $= \prod_{\substack{\text{all links along} \\ \text{path}}} \sigma_{z,ij}$

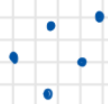
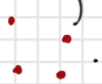
changes charge at terminal points. If system contains 'chemical potential' $\mu \cdot \sum_i \hat{p}_i$, energy cost 2μ.

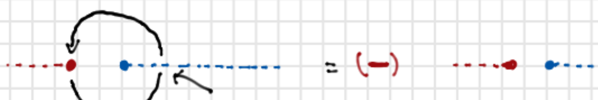
Think of e, m as quantum particles forming on top of (closed) string ground state

Q: What is the statistics of these particles?

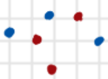
A: Find out what happens as we braid them around one other.



$\rightsquigarrow$ nothing happens  : a system of bosons. The same with .

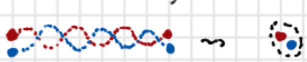
However: 

string of σ_x 'cuts' through one of σ_z: a minus sign

 are semions relative to each other. Note: particle exchange: a π-rotation

	π-exch.	2π-braid
fermions	-1	1
semions	i	-1

Something interesting happens if we fuse m and e into a composite excitation:

 are fermions relative to each other. Have generated fermions

as
- emergent particles of gauge theory
- terminal excitations of strings (cf. Jordan-Wigner in 1d)
- particles obeying strict parity conservation

$\Bigg\} \rightsquigarrow$ conceptual proximity to string theory

- Summary:
  - $\mathbb{Z}_2$ gauge theory has phase transition without local order parameter or symmetry breaking
  - $\mathbb{Z}_2$ topological fluid = a string condensate
  - Rigorous ($L\to\infty$) ground state degeneracy 2^3 (the closest approx. to an 'order parameter')
  - Supports (abelian) quasi particles as excitations, including emergent fermions

- Appendix: U(1) lattice gauge theory (ideas)

Put electrodynamics on a 3+1 dim. lattice. Ingredients (Euclidean metric)

- gauge potential A_{ij} sits on links and enters through phases $e^{iA_{ij}}$
- gauge transformation ϕ_i acts on nodes via phases $e^{i\phi_i}$
- gauge invariant action: $S[A] \sim \sum_{\square} e^{iA_{ij}} e^{iA_{jk}} e^{iA_{kl}} e^{iA_{li}}$

  becomes $S_E[A] = \frac{1}{4}\int d^4x \; F_{\mu\nu}F^{\nu\mu}$ $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in continuum limit (and passage to Minkowski)
- Hamiltonian of lattice gauge theory

  Chose gauge $A_0 = \varphi = 0$. $S_e[A] = \frac{1}{2}\int d^4x \left(\partial_t A_i \partial_t A_i - (\nabla\times A)_i (\nabla\times A_i)\right) =$

  $$\left(= \frac{1}{2}\int d^4x \left(E_i E_i - B_i B_i\right)\right)$$

  Canonical momentum (of 'coordinate' A) $\delta S_e[A] / \delta\, \partial_t A_i = E_i$

  $\rightsquigarrow$ Hamiltonian $H[A,E] = \frac{1}{2}\int d^4x \left(E_i^2 + B_i^2\right) \rightsquigarrow$ on a 3d (!) lattice:

  $$H = \sum_{\langle i,j\rangle} \hat{E}_{ij}^2 + \sum_{\square} e^{i\hat{A}_{ij}} \cdots e^{i\hat{A}_{li}} \qquad [\hat{A}_{ij}, \hat{E}_{ij}] = 1$$

  this is the U(1) analog of the $\mathbb{Z}_2$ - Hamiltonian discussed above.