Scaling theory of Anderson localization

System: (i) free electrons in disorder potential;
(ii) dimension $d = 1, 2, 3$

Model: $\hat{H} = \hat{H}_0 + V(\vec{r})$

(i) $H_0 = p^2/2m$ (or any periodic crystal potential)

(ii) $V(\vec{r}) \to$ random potential

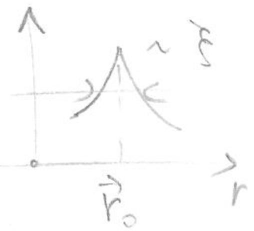
- Weak disorder / classical limit: $(d=3)$

$\sigma_0 = \frac{ne^2\tau}{m}$ (*) – "Drude conductivity" | $\tau \to$ elastic scat. time; $n \to$ concentration

↓ follows from Boltzman kinetic eq

- Strong disorder: $\tau^{-1} \sim \varepsilon_F \to$ Fermi energy

$|\psi(\vec{r})|^2 \sim \exp\left(-|\vec{r}-\vec{r}_0|/\xi\right) \to$ wave function is localized



$\xi \to$ localization length

(Anderson '58)

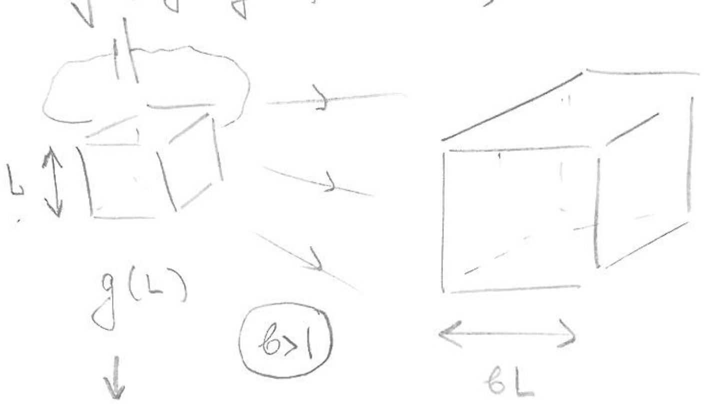
! $d=1$: arbitrary weak disorder $\to$ localization
(Mott & Twose '61, Berezinsky '73 ... many others)

(*) Q: What is τ? Let $V(\vec{r}) = \sum_i V_{imp}(\vec{r}-\vec{r}_i)$ ↗ impurity potential

$$\frac{1}{\tau} = 2\pi n_{imp} \int_0^{\pi} \sigma(\theta)(1-\cos\theta)\, d\theta$$

↓ $\sigma(\theta)$: cross-section at $\varepsilon = \varepsilon_F$

- Scaling theory ("gang of four") (Abrahams, Anderson, Licciardello & Ramakrishnan '79)



$g(L)$ ↓ conductance of cube L^d

Q: What is $g(bL)$?

A: $g(bL) = f(b, g(L))$

i.e. knowledge of $g(L)$ provides the answer for larger block & no any other details do matter!

$\Rightarrow \frac{\partial}{\partial b}(\dots)\Big|_{b\to 1}$ gives $\boxed{\frac{d\ln g(L)}{d\ln L} = \beta(g)}$, where

$\beta(g) \equiv \frac{1}{g} f'_b(1, g)$

A. "Metalic" phase: $g \gg 1$

wire → 

~ A → cross-section

$g \sim \sigma_0 A/L$; $A \sim L^{d-1}$

(g → conductance; σ_0 → conductivity)

$g(L) = \sigma_0 L^{d-2}$

$\Rightarrow \beta(g) = d-2$

(indeed, $\ln g = (d-2)\ln L + \ln \sigma_0$)

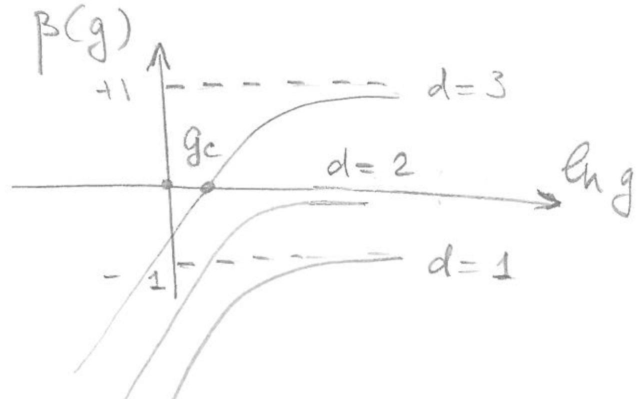
- $I = gV$
- $\vec{j} = \sigma \vec{E}$

B. "Localized" phase: $g(L) = g_0 e^{-L/\xi}$, $g \ll 1$

$\Rightarrow$ (↑ exercise) $\beta(g) = \ln(g/g_0)$

N.B. Physical conductance: $G = \frac{e^2}{2\pi\hbar} g$,

$\frac{e^2}{2\pi\hbar} \equiv G_Q = 12{,}5\ k\Omega^{-1}$ → conductance quantum

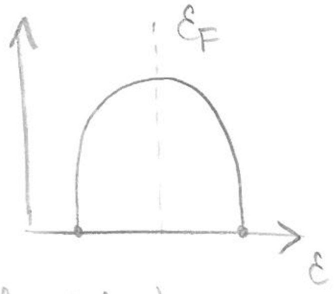


- $d \leq 2 \to$ all states are localized at $L \to \infty$
- $d > 2 \to$ there is unstable fixed point g_c

Fig:
AALR '79 → **one** parameter (g) scaling

- "Mobility edge"

d=3



DoS of ideal periodic crystal

add disorder →

$E_c \to$ mobility edge

localized states

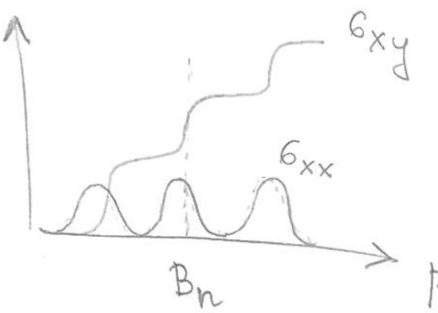
a) $\xi(E) \propto |E - E_c|^{-\nu}$ → localization length close to E_c (ν → crit. exponent)

(b) $\sigma \propto (E - E_c)^s$ → conductivity above E_c

$$\boxed{s = \nu(d-2)}$$ (Wegner '76).

By changing ε_F (concentration) one may drive the system via QCP ("quantum critical point")

- IQHE → one needs **two** parameter scaling



B → magnetic field

$\xi(B) \propto |B - B_n|^{-\nu}$

(to be discussed later on)

Field-theoretical description

- Hamiltonian: $\hat{H} = H_0 + \hat{V}(\vec{r})$

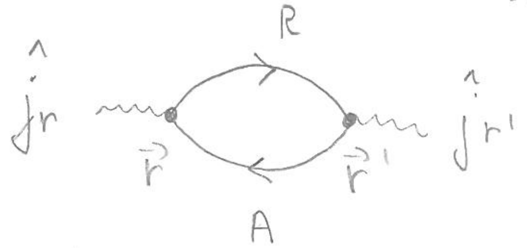
$$\langle V(\vec{r})\, V(\vec{r}\,')\rangle_{dis} = \gamma\, \delta(\vec{r}-\vec{r}\,')$$

↓ "white" noise disorder

(via Kubo formula) Green's function ↓

- Conductivity: $$\bar{\sigma}^{(\vec{r},\vec{r}\,')}_{xx} = \frac{e^2}{2\pi m^2}\left\langle G^A(\vec{r},\vec{r}\,')\, \overleftrightarrow{\nabla}_r \overleftrightarrow{\nabla}_{r'}\, G^R(\vec{r}\,',\vec{r})\right\rangle_{dis}$$

$$f \overleftrightarrow{\nabla}_r g := f(\vec{\nabla} g) - (\vec{\nabla} f) g$$



$\hat{j}(\vec{r}) = -\frac{ie}{m} \overleftrightarrow{\nabla}_r$ → current operator

$$\bar{j}_x(r) = -\int dr'\, \bar{\sigma}_{xx}(\vec{r},\vec{r}\,')\, \nabla_x \varphi(r')\, dr' \quad \Big| \quad G^{R/A} = \left(\varepsilon - H \pm i\delta\right)^{-1}$$

↙ non-local conductivity; ↓ electrostatic potential

- Action

(a) $\mathcal{Z}[\hat{a}] = \int \mathcal{D}(\psi,\bar{\psi})\, e^{-S[\psi,\bar{\psi},\hat{a}]}$ → partition sum

↓ source vector potential

(b) $$S[\psi,\bar{\psi},a] = -i\int dx\, \bar{\psi}\left(\varepsilon + i\delta\tau_3 - \hat{H}_0(\hat{a}) - V\right)\psi$$

$$H_0(\hat{a}) = H_0 \big|_{\hat{p}\to\hat{p}-ie\hat{a}} \quad \text{with } \hat{a} \propto \hat{\tau}_1 a(x)$$

R/A space: $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$, $\bar{\psi} = (\bar{\psi}_+, \bar{\psi}_-)$

(c) $$\sigma_{xx}(\vec{r},\vec{r}\,') = \frac{\delta}{\delta a_x(\vec{r})}\frac{\delta}{\delta a_x(\vec{r}\,')} \ln \mathcal{Z}[\hat{a}]\Big|_{a\to 0}$$

↙ conductivity for given realization of disorder

1.5

Q: How to average $\langle \ln Z[a] \rangle_{dis}$?

- Replica trick:

A: $$\langle \ln Z \rangle_{dis} = \langle \lim_{R\to 0} \left(\frac{Z^R - 1}{R} \right) \rangle_{dis}$$

=> strategy: take $R \in \mathbb{Z}$ → then send to zero in the very end of calculations (via analyt. cont.)

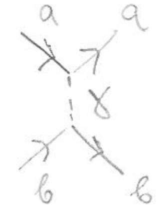
ⓐ Replicated action: $\psi \to \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, where

$\psi_\sigma = (\psi_1^\sigma, \dots, \psi_R^\sigma)$ → replicas

$$Z^R[a] = \int \mathcal{D}(\psi,\bar\psi)\, e^{-S[\psi,\bar\psi]} = Z_1[a] \dots Z_R[a] = Z_1^R[a]$$

ⓑ Averaged partition sum can be easily found:

$$\langle Z^R[a] \rangle_{dis} = \int \mathcal{D}(\psi,\bar\psi)\, e^{-S_0[\psi,\bar\psi] - S_{dis}[\psi,\bar\psi]}$$



(1) $$S_{dis}[\psi,\bar\psi] \overset{*}{=} \frac{\gamma}{2} \sum_{ab} \int dx\, \bar\psi^a(x)\psi^a(x)\bar\psi^b(x)\psi^b(x)$$

$a = (r, \sigma)$, $r = 1, \dots, R$; $\sigma = \pm$ → combined index

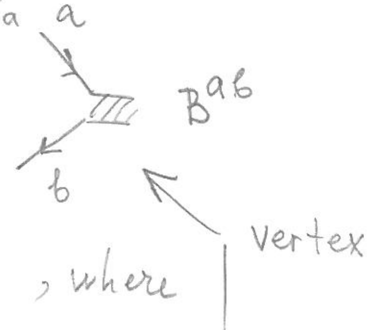
(*) I have used that $\langle e^{-iV} \rangle_{dis} = e^{-\frac{1}{2}\langle V^2 \rangle_{dis}}$ for Gaussian distr. random field.

ⓒ Replica rotation symmetry (in the limit $\delta \to 0$)

$\psi \to U\psi$, where $U(2R)$ → unitary matrix in the $(RA \otimes \text{replicas})$ space

5a|

- Derivation of effective field theory



ⓐ Step # 1 : H.S. decoupling

$$\langle Z^R[a]\rangle = \int \mathcal{D}(\psi,\bar\psi)\int \mathcal{D}B\, e^{-S[\psi,\bar\psi,B]}$$, where

$$S[\psi,\bar\psi,B] = \frac{1}{2\gamma}\int \mathrm{tr}\, B^2\, dx - i\int dx\, \bar\psi(\varepsilon + i\delta\hat\tau_3 - H_0 - iB)\psi$$

Here $B^{2R\times 2R}(x)$ is matrix field in RA⊗ replica space

Gaussian int. over B gives back original averaged action (1)

ⓑ. Step # 2: Integrate out ψ & $\bar\psi$ and find saddle point over B:

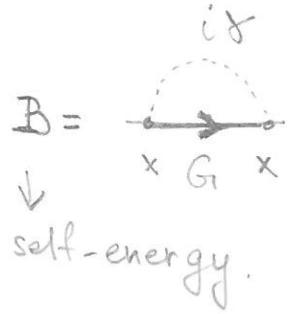
(i) $S_{eff}[B] = \frac{1}{2\gamma}\int \mathrm{tr}\, B^2(x)\, dx - \ln\det(\varepsilon + i\delta\hat\tau_3 - H_0 - iB)$

(ii) Saddle point: $\frac{\delta S_{eff}}{\delta B} = 0 \Rightarrow$ $\boxed{B(x) = i\gamma\, \hat G[B]\,\big|_{x=x'}}$

where $G[B](x,x') := \left(\varepsilon + i\delta\hat\tau_3 - H_0 - iB\right)^{-1}(x,x')$.

S.p. eq. is known as self-consistent Born approximation (SCBA)

1) "Physical" solution: $\bar B = \frac{1}{2\tau}\hat\tau_3 \otimes \mathbb{1}$ (↑ replicas)



$1/2\tau = 2\pi\nu\gamma$, $\nu \to$ Density of states

! $\bar B$ breaks $U(2R)$ symmerty of original action (sponteneous symmetry breaking at $\delta \to 0^+$).

2) Other saddle points $\bar B = \frac{1}{2\tau} T\hat\tau_3 T^{-1}$; $T \in U(2R)$

ⓒ Step # 3 : $T \to T(x)$ & expand the action in soft modes (Similar to G.-L. action)

L.6 • Field theory (long-wave limit) : "σ-model" ↓ historical name

$$\langle \mathcal{Z}^R[a] \rangle \overset{!}{=} \int \mathcal{D}\hat{Q}\, e^{-S[\hat{Q}, a]} \quad \text{, where}$$

ⓐ $\hat{Q} = \hat{T}\hat{\tau}_3\hat{T}^{-1}$ → matrix field, $Q^{2R\times 2R}$,

where $T \in \mathcal{U}(2R)$ → unitary rotation,

$Q \in \mathcal{U}(2R)/\mathcal{U}(R)\otimes\mathcal{U}(R) = G/H$ → coset space (spont. symmetry breaking)

$H = \left\{ \begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_2 \end{pmatrix} \middle| \ \mathcal{U}_i \in \mathcal{U}(R) \right\}$ → invariant subgroup

ⓑ Low-energy action : $S[\hat{Q}] = \frac{1}{8} g_{xx} \int dx \, \mathrm{tr}\left(\vec{\nabla}_\mu Q \vec{\nabla}_\mu Q\right)$ (↗ Drude conductance)

(i) $g_{xx} = 2\pi\nu D$ (ν → DoS, $D = \frac{1}{d} v_F^2 \tau$ → diffusion constant)

(ii) $\vec{\nabla}_\mu Q := \partial_\mu Q - i[\hat{a}_\mu, Q]$ → covariant (long) derivative

(remember, $\hat{a}_\mu \propto \tau_1 a_\mu$ has matrix structure!)

∂_μ with $\mu = 1, .., d$ → conventional derivative

ⓒ Matrix field describes bilinears:

$Q^{rr'}_{\sigma\sigma'}(x) \sim \psi^r_\sigma(x)\bar{\psi}^{r'}_{\sigma'}(x)$ | $r = 1, \dots, R$ → replicas; $\sigma = \pm$, R/A index

ⓓ See next page →

$\hat{Q} \leftrightarrow \vec{m}$

(d) <u>σ-model</u> of disordered metal ↔ spin chain (with S = even)

$$S = \frac{1}{\lambda}\int_0^\beta d\tau \int_0^L dx \left((\partial_x \vec m)^2 + (\partial_\tau \vec m)^2\right), \quad v_s = 1$$

$\vec m \in S^2 \cong SU(2)/U(1)$ sphere

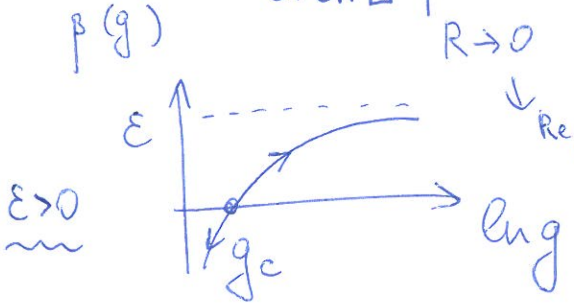
- $\vec m^2 = 1$ & $Q^2 = (T\sigma_3 T^{-1})^2 = \mathbb{1}^{2R\times 2R}$

(e) <u>σ-model</u> is renormalizable at $g \gg 1$

$[g_{xx}] = D-2 = \varepsilon$ → $D=2$ is critical dimension

$$\left.\frac{d\ln g}{d\ln L}\right|_{R\to 0} = \varepsilon - \frac{1}{2\pi^2 g^2} + O(g^{-4}) = \beta(g)$$

↓ Replica limit



Wegner '89

$g_c = \frac{1}{\pi}\frac{1}{\sqrt{2\varepsilon}}$ | $s = \nu(d-2)$

$\nu^{-1} = g_c \beta'(g_c) \Rightarrow$ $\boxed{\nu = 1/2\varepsilon \Rightarrow s = \frac{1}{2}}$

- Justification of single-parameter scaling by AALK

(f) IQHE: extra (topological) term in the action (like in spin chain with S = odd)

(g_{xx}, g_{xy}) → two parameters (to be continued...)