Musterlösung - Teil A -

1) a) $\|v\| = \sqrt{\langle v,v\rangle}$ 1

b) $\langle e_1', e_1'\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\cdot(1+1) = 1$

$\langle e_2', e_2'\rangle = \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix} = 1$ 1

$\langle e_1', e_2'\rangle = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix} = 0$

e_1', e_2' sind keine Vielfachen voneinander 1

$\Rightarrow$ $\mathcal{B}'$ ist ONB

c) $\boxed{v_1' = \langle v, e_1'\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-5\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \frac{-3}{\sqrt{2}}$

1 $\boxed{v_2' = \langle v, e_2'\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix} = \frac{7}{\sqrt{2}}$

$\Rightarrow v = \frac{1}{\sqrt{2}}\begin{pmatrix}-3\\7\end{pmatrix}_{\mathcal{B}'}$ 1

2) $V^* = \{ f : V \to \mathbb{R} \mid f \text{ ist linear} \}$ 1

$f \oplus g$ und $\lambda \odot f$ werden punktweise definiert 1

3) a) $\dim U = 2$ $\dim V = 3$ 1

b) 1

$\boxed{L(e_1)} = (\underbrace{\vartheta_1(e_1)}_{1} - \underbrace{\vartheta_2(e_1)}_{0})\, f_1 + 3\underbrace{\vartheta_2(e_1)}_{0} f_3 = f_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}_V}$

$\boxed{L(e_2)} = (\underbrace{\vartheta_1(e_2)}_{0} - \underbrace{\vartheta_2(e_2)}_{1})\, f_1 + 3\underbrace{\vartheta_2(e_2)}_{1} f_3 = -f_1 + 3f_3 = \begin{pmatrix}-1\\0\\3\end{pmatrix}_{\mathcal{B}_V}$

$\Rightarrow M(L) = \begin{pmatrix}1 & -1\\0 & 0\\0 & 3\end{pmatrix}$ 1 hierfür gibt es noch keinen Punkt, da dies explizit auf dem Infozettel stand!

4) a) $\overbrace{(D_p g)(v) = \frac{d}{dt}\Big|_{t=0} g(p+tv)} = \frac{d}{dt}\Big|_{t=0} \exp(x(p+tv) + y(p+tv))$

$$= \frac{d}{dt}\Big|_{t=0} \exp(x(p) + tv_x + y(p) + tv_y)$$

$$= (v_x + v_y) \exp(x(p) + y(p)) \quad 1$$

$$\Rightarrow D_p g = \exp(x(p) + y(p))(dx + dy) \quad 1$$

b) $(g \circ f)(t) = \exp(t + t^2)$ 1

c) $(g \circ f)'(t) = (1 + 2t) \exp(t + t^2)$ 1

d) $J f = \begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \quad J g = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \exp(x+y) \begin{pmatrix} 1 & 1 \end{pmatrix}$ 1

$$J_{f(p)} g = \exp(t + t^2) \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$J_p (g \circ f) = \exp(t + t^2) \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2t \end{pmatrix} = \exp(t + t^2)(1 + 2t) \quad 1$$

5) $d(xy) = x\,dy + y\,dx$ 1 $\quad d(z^2 dx) = 2z\,dz \wedge dx$ 1

$d((x^2 + y^2)\, dx \wedge dz) = 2y\, dy \wedge dx \wedge dz$ 1

6) a) $\gamma : [0,1] \to \mathbb{E}_2 \; ; \; t \mapsto p_0 + t(2e_x + e_y)$ 1

b) $\gamma' = 2e_x + e_y$

$$(x\,dx + y\,dy)_{\gamma(t)}(\gamma'(t)) = 2t \cdot 2 + t \cdot 1 = 5t$$

$$\int_\gamma x\,dx + y\,dy = \int_0^1 5t\,dt = 5 \frac{t^2}{2}\Big|_0^1 = \frac{5}{2} \quad 1$$

c) $\Phi = \frac{1}{2}(x^2 + y^2)$ ist ein Potential 1

$$\int_\gamma x\,dx + y\,dy = \Phi(\gamma(1)) - \Phi(\gamma(0)) = \Phi(p_0 + 2e_x + e_y) - \Phi(p_0)$$

$$= \frac{1}{2}(2^2 + 1^2) = \frac{5}{2} \quad 1$$

7) a) $\sigma : [0,2] \times [0,2\pi] \to \mathbb{E}_2$

$(r, \varphi) \mapsto p_0 + r(\cos\varphi\, e_x + \sin\varphi\, e_y)$ (1)

b) $\frac{\partial\sigma}{\partial r} = \cos\varphi\, e_x + \sin\varphi\, e_y \qquad \frac{\partial\sigma}{\partial r} = r(-\sin\varphi\, e_x + \cos\varphi\, e_y)$

$$\left((x^2+y^2)\, dx\wedge dy\right)_{\sigma(r,\varphi)}\left(\frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial\varphi}\right) = r^2 \begin{vmatrix} dx(\cos\varphi\, e_x + \sin\varphi\, e_y) & dx(-r\sin\varphi\, e_x + r\cos\varphi\, e_y) \\ dy(\text{— " —}) & dy(\text{— " —}) \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r^3 \quad (1)$$

$$\int_\Sigma (x^2+y^2)\, dx\wedge dy = \int_0^{2\pi} d\varphi \int_0^2 dr\, r^3 = 2\pi\, \tfrac{1}{4} r^4 \Big|_0^2 = 2\pi \cdot 4 = 8\pi \quad (1)$$

8) $\int_\Sigma d\omega = \int_{\partial\Sigma} \omega$ (1)

9) a) $\int \frac{dy}{y} = \int 2x\, dx \rightsquigarrow \ln|y| = x^2 + \tilde{c} \rightsquigarrow y(x) = c e^{x^2}$ (1)

$y(0) = c = 1$

b) Ansatz $y = e^{\lambda x} \rightsquigarrow \lambda^2 + \lambda - 2 = 0 \Leftrightarrow \lambda_1 = 1 \vee \lambda_2 = -2$

$\rightsquigarrow y(x) = c_1 e^x + c_2 e^{-2x}$ (1)

$$\left.\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 - 2c_2 = 3 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = -1 \quad (1)$$

– Teil B –

10) a) $h : [0,R] \times [0,2\pi] \times [-\frac{L}{2}, \frac{L}{2}] \to \mathbb{E}_3$

$(r,\varphi,z) \mapsto p_0 + r(\cos\varphi\, e_x + \sin\varphi\, e_y) + z e_z$ (1)

b) Eine „normale" 3-Form hat die Eigenschaft, unter Vertauschung der Orientierung der Parametrisierung das Vorzeichen zu wechseln; die Masse $\int_U \rho$ soll jedoch stets positiv sein. (1)

c) Tangentialvektoren:

$\frac{\partial h}{\partial r} = \cos\varphi\, e_x + \sin\varphi\, e_y \quad \frac{\partial h}{\partial \varphi} = -r\sin\varphi\, e_x + r\cos\varphi\, e_y \quad \frac{\partial h}{\partial z} = e_z$ (1)

$[dx \wedge dy \wedge dz, R]$ auswerten:

$$\begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \quad \text{(oder ähnlich...)}$$ (1)

$$\int_{[U,R]} \rho\,[dx\wedge dy\wedge dz, R] = \rho_0 \int_0^R r\,dr \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{\frac{L}{2}} dz = \pi R^2 L \rho_0 = M$$ (1)

d) $$\int_{[U,R]} (x^2+y^2)\,[dx\wedge dy\wedge dz, R]$$

$$= \rho_0 \int_0^R r^3 dr \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{\frac{L}{2}} dz$$ (1)

$$= \rho_0 \frac{1}{4} R^4 \cdot 2\pi L$$ (1)

$$= \frac{1}{2} M R^2$$ (optional)

11) a) $\partial_u = \frac{\partial x}{\partial u} e_x + \frac{\partial y}{\partial u} e_y = u e_x + v e_y$

$\partial_v = \frac{\partial x}{\partial v} e_x + \frac{\partial y}{\partial v} e_y = -v e_x + u e_y$ (1)

$\langle \partial_u, \partial_v \rangle = -uv + uv = 0$ (1)

b) $\|\partial_u\| = \|\partial_v\| = \sqrt{u^2+v^2}$ (1)

$\Rightarrow \hat{e}_u = \frac{1}{\|\partial_u\|} \partial_u \quad \hat{e}_v = \frac{1}{\|\partial_v\|} \partial_v$

c) du, dv bilden (punktweise) eine Basis

$\Rightarrow \exists a, b, c, d$ (reellwertige Funktionen) $\mathcal{I}(\hat{e}_u) = a\,du + b\,dv$

$\mathcal{I}(\hat{e}_v) = c\,du + d\,dv$ (1)

$\mathcal{I}(\hat{e}_u) \overset{(\partial_u)}{=} \langle \hat{e}_u, \partial_u \rangle = \|\partial_u\| = a$

$\mathcal{I}(\hat{e}_u)(\partial_v) = 0 = b$

$\mathcal{I}(\hat{e}_v)(\partial_u) = 0 = c$

$\mathcal{I}(\hat{e}_v)(\partial_v) = \|\partial_v\| = d$ (1)

d) $du = \frac{1}{\|\partial_u\|} \mathcal{I}(\hat{e}_u) \quad dv = \frac{1}{\|\partial_v\|} \mathcal{I}(\hat{e}_v)$

$\Rightarrow \mathcal{I}^{-1}(du) = \frac{1}{\|\partial_u\|} \hat{e}_u \quad \mathcal{I}^{-1}(dv) = \frac{1}{\|\partial_v\|} \hat{e}_v$ (1)

$\operatorname{grad} f = \mathcal{I}^{-1}(df) = \frac{1}{\|\partial_u\|} \frac{\partial f}{\partial u} \hat{e}_u + \frac{1}{\|\partial_v\|} \frac{\partial f}{\partial v} \hat{e}_v$ (1)

12) a) $\partial \boxed{\circlearrowleft} = \leftarrow + \downarrow + \rightarrow + \uparrow$ (1)

$\partial \rightarrow = \bar{\cdot} \quad \cdot^{+}$

b) $\partial^2 \boxed{\circlearrowleft} = \ldots = 0$ (1)

c) Basiselemente $\cdot^{R}/_{L}$ bzw

bzw

nicht unbedingt erforderlich

Paarung

$\langle \cdot , \boxed{\;} \rangle = +1 = -\langle \cdot , \boxed{\;} \rangle$

$\langle \uparrow , \uparrow \rangle = +1 = -\langle \downarrow , \uparrow \rangle$ (2)

$\langle \blacksquare , \cdot^{\pm} \rangle = \pm 1$

bzw $= 0$, falls Punkt außerhalb des Quadrats/ Linien sich nicht schneiden !

d) Stokes $\int_K d\omega = \int_{\partial K} \omega$

$1 = \boxed{\;} \overset{!}{=}$ $\Rightarrow$ Das Linienstück muss $\dashv$ sein (1)

$\Rightarrow d \dashv =$

$1 = \bar{\cdot}$ $\blacksquare$ $=$ $\Rightarrow d \blacksquare =$ (1)

$d^2 \blacksquare = d \boxed{\;} = \ldots + \ldots + \ldots + \ldots = 0$ (1)

13) a) Für $a, b \neq 0$ hat man $(f, g \in M)$

$$(f+g)(0) = 2a \neq a \qquad (f+g)(1) = 2b \neq b \qquad 1$$

$$\Rightarrow f+g \notin M$$

b) Die Differenz zweier Funktionen muss in V liegen, und es ist stets $(f, g \in M)$

$$(f-g)(0) = a - a = 0 \qquad (f-g)(1) = b - b = 0 \qquad 1$$

c) $$(D_f S)(g) \underset{1}{=} \frac{d}{dt}\Big|_{t=0} \int_0^1 L\big(f(x) + t g(x),\ f'(x) + t g'(x)\big)\, dx$$

Korrektes Einsetzen!

$$\underset{2}{=} \int_0^1 \left\{ \frac{\partial L}{\partial y_1}\big(f(x), f'(x)\big)\, g(x) + \frac{\partial L}{\partial y_2}\big(f(x), f'(x)\big)\, g'(x) \right\} dx$$

(Hier sollte die Kettenregel ~~erwähnt~~ verwendet werden)

d) $$\underset{1}{=} \underbrace{\frac{\partial L}{\partial y_2}\big(f(x), f'(x)\big)\, g(x)\Big|_{x=0}^{x=1}}_{=0 \quad 1}$$

$$+ \int_0^1 \left\{ \frac{\partial L}{\partial y_1}\big(f(x), f'(x)\big) - \frac{d}{dx} \frac{\partial L}{\partial y_2}\big(f(x), f'(x)\big) \right\} g(x)\, dx$$