Musterlösung Teil A

1) a) $e_1' \cdot e_2' = \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\end{pmatrix} = 0$

$e_1' \cdot e_1' = \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix} = 1 = e_2' \cdot e_2'$ ②

b) $\begin{pmatrix}2\\-5\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix} = \frac{-1}{\sqrt{5}}$ $\quad \begin{pmatrix}2\\-5\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix} = \frac{12}{\sqrt{5}}$

$\Rightarrow \begin{pmatrix}2\\-5\end{pmatrix}_B = \frac{1}{\sqrt{5}}\begin{pmatrix}-1\\12\end{pmatrix}_{B'}$ ②

2) $f\begin{pmatrix}x+\lambda x'\\y+\lambda y'\end{pmatrix} = \begin{pmatrix}x+y+\lambda(x'+y')\\x+\lambda x'\end{pmatrix} = f\begin{pmatrix}x\\y\end{pmatrix} + \lambda f\begin{pmatrix}x'\\y'\end{pmatrix} \Rightarrow$ linear ①

$g\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow$ nicht linear ①

$h\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix} \neq 2\cdot h\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow$ nicht linear ①

3) a) $Le_1 = 5f_1 - 2f_2 = \begin{pmatrix}5\\-2\\0\end{pmatrix}_{B_V}$ $\quad Le_2 = -f_1 + 3f_3 = \begin{pmatrix}-1\\0\\3\end{pmatrix}_{B_V}$

$\Rightarrow M(L) = \begin{pmatrix}5 & -1\\-2 & 0\\0 & 3\end{pmatrix}$ ②

b) $L^t(\varphi_1) = 5\vartheta_1 - \vartheta_2 = \begin{pmatrix}5\\-1\end{pmatrix}_{B_V^*}$ $\quad L^t(\varphi_2) = -2\vartheta_1 = \begin{pmatrix}-2\\0\end{pmatrix}_{B_V^*}$

$L^t(\varphi_3) = 3\vartheta_2 = \begin{pmatrix}0\\3\end{pmatrix}_{B_V^*} \Rightarrow M(L^t) = \begin{pmatrix}5 & -2 & 0\\-1 & 0 & 3\end{pmatrix} = M(L)^t$ ②

4) a) $\frac{d}{dt}\, g(p + t(v_x e_x + v_y e_y)) = (2v_x - v_y)\exp(2x-y)$ ①

b) $\Rightarrow Dg = \exp(2x-y)\,(2dx - dy)$ ①

c) $(g\circ f)(t) = \exp(2t^2 - t^3)$ ① $\quad$ d) $(g\circ f)'(t) = (4t - 3t^2)\exp(2t^2 - t^3)$ ①

d) $J_g = \begin{pmatrix}2\exp(2x-y) & -\exp(2x-y)\end{pmatrix}$ $\quad J_f = \begin{pmatrix}2t\\3t^2\end{pmatrix}$ ①

$\Rightarrow J_{g\circ f} = J_{g}\circ f \cdot J_f = \exp(2t^2 - t^3)\underbrace{\begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}2t\\3t^2\end{pmatrix}}_{4t-7t^2}$ ①

5) a) $d(y^2 z) = 2yz\,dy + y^2 dz$ (1)

$d(\ldots) = x\, dy \wedge dx \wedge dz$ (1)

6) a) $\gamma : [0,1] \to \mathbb{E}_3\ ;\ t \mapsto p_0 + t e_x + 3(t-1) e_y$ (1)

b) $\Phi = \frac{1}{3} x^3 + \frac{1}{3} y^3$ (1)

$\Phi(p_0 + e_x) - \Phi(p_0 - 3e_y) = \frac{1}{3} - (-9) = \frac{28}{3}$ (1)

7) a) $\sigma : [0,3] \times [0,2\pi] \to \mathbb{E}_2\ ;\ (r,\varphi) \mapsto p_0 + r(\cos\varphi\, e_x + \sin\varphi\, e_y)$ (1)

b) $\frac{\partial\sigma}{\partial r} = \cos\varphi\, e_x + \sin\varphi\, e_y \qquad \frac{\partial\sigma}{\partial\varphi} = -r\sin\varphi\, e_x + r\cos\varphi\, e_y$

$$\int_\Sigma (x^2+y^2)\, dx \wedge dy = \int_0^3 dr \int_0^{2\pi} d\varphi\, r^2 \cdot r = \frac{81}{4} \cdot 2\pi$$ (2)

c) Rand $\partial\Sigma : [0,2\pi] \to \mathbb{E}_2\ ;\ \varphi \mapsto p_0 + 3(\cos\varphi\, e_x + \sin\varphi\, e_y)$ (1)

$$\int_\Sigma \cdots = \int_{\partial\Sigma} (-x^2 y\, dx + x^2 y^2\, dy)$$

$$= \int_0^{2\pi} 2 \cdot 81 \cos^2\varphi \sin^2\varphi\, d\varphi = \frac{81}{4}\left(x - \frac{1}{4}\sin 4x\right)\Big|_0^{2\pi} = \frac{81}{4} \cdot 2\pi.$$ (1)

8) a) $\int \frac{dy}{y} = 3\int x^2 dx \Rightarrow y(x) = c\, e^{x^3} \quad y(0) = c \overset{!}{=} 1$ (1)

b) char. Polynom $\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$

$\Rightarrow y(x) = c_1 e^x + c_2 x e^x$ (1)

$y(0) = c_1 \overset{!}{=} 1 \qquad y'(0) = c_1 + c_2 \overset{!}{=} 0 \Rightarrow c_2 = -1$ (1)

Musterlösung Teil B

9) a) $\sigma : [R_1, R_2] \times [0, 2\pi] \times [-\frac{L}{2}, \frac{L}{2}] \to \mathbb{E}_3$

$$(r, \varphi, z) \mapsto p_0 + r(\cos\varphi\, e_x + \sin\varphi\, e_y) + z e_z \quad (1)$$

(rechtshändig orientiert)

b)
$$\frac{\partial\sigma}{\partial r} = \cos\varphi\, e_x + \sin\varphi\, e_y$$
$$\frac{\partial\sigma}{\partial\varphi} = -r\sin\varphi\, e_x + r\cos\varphi\, e_y \quad (1)$$
$$\frac{\partial\sigma}{\partial z} = e_z$$

$$(dx \wedge dy \wedge dz)\left(\frac{\partial\sigma}{\partial r}, \frac{\partial\sigma}{\partial\varphi}, \frac{\partial\sigma}{\partial z}\right) = \begin{vmatrix} \cos\varphi & \sin\varphi & 0 \\ -r\sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \quad (1)$$

$$\int_{[U,R]} \rho = \rho_0 \int_{R_1}^{R_2} dr \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{\frac{L}{2}} dz\; r = 2\pi L \rho_0 \int_{R_1}^{R_2} r\, dr = \pi L \rho_0 (R_2^2 - R_1^2) \quad (1)$$

c) $$\int_{[U,R]} (x^2+y^2)\rho = 2\pi L \rho_0 \int_{R_1}^{R_2} r^3\, dr = \frac{1}{2}\pi L \rho_0 (R_2^4 - R_1^4) = \frac{1}{2} M (R_2^2 + R_1^2) \quad (1)$$

d) $I_z \to M R_2^2$ (1)

e) Die Masse ist hier weiter von der Drehachse entfernt, I_z also größer. (1)

10) a)

$$ma = m\dot{v} \overset{\uparrow}{=} F = -mg \overset{\downarrow}{-} \beta v$$

v<0, negatives Vorzeichen führt zu geringerer Beschleunigung

"Newton" (unter =), Gewicht (unter $-mg$), Reibung (unter $-\beta v$)

(2)

b) $\dot{v} \xrightarrow[t\to\infty]{} 0 \Rightarrow v_\infty = \frac{-mg}{\beta}$ (1)

c) homogene DGL $\dot{v} = -\frac{\beta}{m} v \Rightarrow v = \tilde{c}\, e^{-\frac{\beta}{m}t}$ (1)

VdK $v = \tilde{c}(t)\, e^{-\frac{\beta}{m}t}$ (1)

$\Rightarrow \dot{v} = \dot{\tilde{c}}\, e^{-\frac{\beta}{m}t} - \frac{\beta}{m} v$

Einsetzen: $\dot{\tilde{c}} = -g e^{\frac{\beta}{m}t}$

$\Rightarrow \tilde{c} = v_\infty e^{\frac{\beta}{m}t} + c$

$\Rightarrow v(t) = v_\infty + c e^{-\frac{\beta}{m}t}$

$v(0) = v_\infty + c \overset{!}{=} v_0 \Leftrightarrow c = v_0 - v_\infty$ (2)

11a) $x = r\cos\varphi \qquad y = r\sin\varphi$

$$\Rightarrow \partial_r = \cos\varphi\, e_x + \sin\varphi\, e_y = \hat{e}_r$$

$$\partial_\varphi = -r\sin\varphi\, e_x + r\cos\varphi\, e_y = r\hat{e}_\varphi \quad ②$$

b) Man betrachtet Kurven $\gamma: I \subset \mathbb{R} \to \mathbb{E}_2$ und meint

$\dot{r} \equiv \frac{d}{dt} r(\gamma(t))$, $\dot{\varphi} \equiv \frac{d}{dt}\varphi(\gamma(t))$. ①

c) $$\dot{\hat{e}}_r = -\dot{\varphi}\sin\varphi\, e_x + \dot{\varphi}\cos\varphi\, e_y = \dot{\varphi}\hat{e}_\varphi \quad ①$$

$$\dot{\hat{e}}_\varphi = -\dot{\varphi}\cos\varphi\, e_x - \dot{\varphi}\sin\varphi\, e_y = -\dot{\varphi}\hat{e}_r \quad ①$$

$$\Rightarrow \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi \quad ①$$

$$\dot{\vec{v}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\varphi}\hat{e}_\varphi + \dot{r}\dot{\varphi}\hat{e}_\varphi + r\ddot{\varphi}\hat{e}_\varphi - r\dot{\varphi}^2\hat{e}_r$$

$$= (\ddot{r} - r\dot{\varphi}^2)\hat{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{e}_\varphi \quad ①$$

12)

$$L(y_1, y_2) = \sqrt{1 + y_2^2} \quad (1)$$

$$\frac{\partial L}{\partial y_1} = 0 \qquad \frac{\partial L}{\partial y_2} = \frac{y_2}{\sqrt{1+y_2^2}} \quad (1)$$

$$\Rightarrow \quad \frac{d}{dx} \frac{f'(x)}{\sqrt{1+f'(x)^2}} = 0 \quad (1)$$

$$\Rightarrow \quad \frac{f'(x)}{\sqrt{1+f'(x)^2}} = \text{const} \quad (1)$$

$$\Rightarrow \quad f'(x) = c \in \mathbb{R} \quad (1)$$

$$\Rightarrow \quad f(x) = cx + d \quad (1)$$

$$\Rightarrow \quad f(x) = \frac{a-b}{x_0 - x_1} x + \frac{b x_0 - a x_1}{x_0 - x_1}$$

Es ist eine Gerade! (1)