Part III

The proceedings of the 1927 Solvay conference

H. A. Lorentz †

Hardly a few months have gone by since the meeting of the fifth physics conference in Brussels, and now I must, in the name of the scientific committee, recall here all that meant to the Solvay International Institute of Physics he who was our chairman and the moving spirit of our meetings. The illustrious teacher and physicist, H. A. Lorentz, was taken away in February 1928 by a sudden illness, when we had just admired, once again, his magnificent intellectual gifts which age was unable to diminish in the least.

Professor Lorentz, of a simple and modest demeanour, nevertheless enjoyed an exceptional authority, thanks to the combination of rare qualities in a harmonious whole. Theoretician with profound views — eminent teacher in the highest forms of instruction and tirelessly devoted to this task — fervent advocate of all international scientific collaboration — he found, wherever he went, a grateful circle of pupils, disciples and those who carried on his work. Ernest Solvay had an unfailing appreciation of this moral and intellectual force, and it was on this that he relied to carry through a plan that was dear to him, that of serving Science by organising conferences composed of a limited number of physicists, gathered together to discuss subjects where the need for new insights is felt with particular intensity. Thus was born the Solvay International Institute of Physics, of which Ernest Solvay followed the beginnings with a touching concern and to which Lorentz devoted a loyal and fruitful activity.

All those who had the honour to be his collaborators know what he was as chairman of these conferences and of the preparatory meetings. His thorough knowledge of physics gave him an overall view of the problems to be examined. His clear judgement, his fair and benevolent spirit guided the scientific committee in the choice of the assistance it was appropriate to call upon. When we then were gathered together at a conference, one could only admire without reservations the mastery with which he conducted the chairmanship. His shining intellect dominated the discussion and followed it also in the details, stimulating it or preventing it from drifting, making sure that all opinions could be usefully expressed, bringing out the final conclusion as far as possible. His perfect knowledge of languages allowed him to interpret, with equal facility, the words uttered by each one. Our chairman appeared to us, in fact, gifted with an invincible youth, in his passion for scientific truth and in the joy he had in comparing opinions, sometimes with a shrewd smile on his face, and even a little mischievousness when confronted with an unforeseen aspect of the question. Respect and affection went to him spontaneously, creating a cordial and friendly atmosphere, which facilitated the common work and increased its efficiency.

True creator of the theoretical edifice that explains optical and electromagnetic phenomena by the exchange of energy between electrons contained in matter and radiation viewed in accordance with Maxwell's theory, Lorentz retained a devotion to this classical theory. All the more remarkable is the flexibility of mind with which he followed the disconcerting evolution of the quantum theory and of the new mechanics.

The impetus that he gave to the Solvay institute will be a memory and an example for the scientific committee. May this volume, faithful report of the work of the recent physics conference, be a tribute to the memory of he who, for the fifth and last time, honoured the conference by his presence and by his guidance.

M. Curie

Fifth physics conference

The fifth of the physics conferences, provided for by article 10 of the statutes of the international institute of physics founded by Ernest Solvay, held its sessions in Brussels on the premises of the institute from 24 to 29 October 1927.

The following took part in the conference:

Mr H. A. LORENTZ †, of Haarlem, *Chairman*.

Mrs P. CURIE, of Paris; Messrs N. BOHR, of Copenhagen; M. BORN, of Göttingen; W. L. BRAGG, of Manchester; L. BRILLOUIN, of Paris; A. H. COMPTON, of Chicago; L.-V. DE BROGLIE, of Paris; P. DEBYE, of Leipzig; P. A. M. DIRAC, of Cambridge; P. EHRENFEST, of Leiden; A. EINSTEIN, of Berlin; R. H. FOWLER, of Cambridge; Ch.-E. GUYE, of Geneva; W. HEISENBERG, of Copenhagen; M. KNUDSEN, of Copenhagen; H. A. KRAMERS, of Utrecht; P. LANGEVIN, of Paris; W. PAULI, of Hamburg; M. PLANCK, of Berlin; O. W. RICHARDSON, of London; [E. SCHRÖDINGER, of Zurich;] C. T. R. WILSON, of Cambridge, *Members*.

Mr J.-E. VERSCHAFFELT, of Gent, fulfilled the duties of *Secretary*.

Messrs Th. DE DONDER, E. HENRIOT and Aug. PICCARD, professors at the University of Brussels, attended the meetings of the conference as guests of the scientific committee, Mr Ed. HERZEN, professor at the École des Hautes Études de Bruxelles, as representative of the Solvay family.

Professor I. LANGMUIR, of Schenectady (U. S. of America), visiting Europe, attended the meetings as a guest.

Mr Edm. van AUBEL, member of the Scientific Committee, and Mr H. DESLANDRES, director of the Meudon observatory, invited to participate in the conference meetings, had been excused.

Sir W. H. BRAGG, member of the scientific committee, who had handed in his resignation before the meetings and requested to be excused, also did not attend the sessions.

The administrative commission of the institute was composed of:

Messrs Jules BORDET, professor at the University of Brussels, appointed by H. M. the King of the Belgians; Armand SOLVAY, engineer, manager of Solvay and Co.; Maurice BOURQUIN, professor at the University of Brussels; Émile HENRIOT, professor at the University of Brussels; Ch. LEFÉBURE, engineer, appointed by the family of Mr Ernest Solvay, *Administrative Secretary.*

The scientific committee was composed of:

Messrs H. A. LORENTZ†, professor at the University of Leiden, *Chairman*; M. KNUDSEN, professor at the University of Copenhagen, *Secretary*; W. H. BRAGG, professor at the University of London, president of the Royal Institution; Mrs Pierre CURIE, professor at the Faculty of Sciences of Paris; Messrs A. EINSTEIN,[a] professor, in Berlin; Charles-Eug. GUYE, professor at the University of Geneva; P. LANGEVIN, professor at the Collège de France, in Paris; O. W. RICHARDSON, professor at the University of London; Edm. van AUBEL, professor at the University of Gent.

Sir W. H. BRAGG, resigning member, was replaced by Mr B. CABRERA, professor at the University of Madrid.

To replace its late chairman, the scientific committee chose Professor P. LANGEVIN.

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[a] Chosen in replacement of Mr H. Kamerlingh Onnes, deceased.

The intensity of X-ray reflection[a]

By Mr W. L. BRAGG

1. — The classical treatment of X-ray diffraction phenomena

The earliest experiments on the diffraction of X-rays by crystals showed that the directions in which the rays were diffracted were governed by the classical laws of optics. Laue's original paper on the diffraction of white[1] radiation by a crystal, and the work which my father and I initiated on the reflection of lines[2] in the X-ray spectrum, were alike based on the laws of optics which hold for the diffraction grating. The high accuracy which has been developed by Siegbahn and others in the realm of X-ray spectroscopy is the best evidence of the truth of these laws. Advance in accuracy has shown the necessity of taking into account the very small refraction of X-rays by the crystal, but this refraction is also determined by the classical laws and provides no exception[3] to the above statement.

The first attempts at crystal analysis showed further that the strength of the diffracted beam was related to the structure of the crystal in a way to be expected by the optical analogy. This has been the basis of most work on the analysis of crystal structure. When monochromatic X-rays are reflected from a set of crystal planes, the orders of reflection are strong, weak, or absent in a way which can be accounted for qualitatively

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[a] We follow Bragg's original English typescript, from the copy in the Richardson collection, AHQP-RDN, document M-0059 (indexed as 'unidentified author' in the microfilmed catalogue). Obvious typos are corrected mostly tacitly and some of the spelling has been harmonised with that used in the rest of the volume. Discrepancies between the original English and the published French are endnoted (*eds.*).

by the arrangement of atoms[4] parallel to these planes. In the analysis of many structures, it is not necessary to make a strict examination of the strength of the diffracted beams. Slight displacements of the atoms cause the intensities of the higher orders to fluctuate so rapidly, that it is possible to fix the atomic positions with high accuracy by using a rough estimate of the relative intensity of the different orders.

When we attack the problem of developing an accurate quantitative theory of intensity of diffraction, many difficulties present themselves. These difficulties are so great, and the interpretation of the experimental results has often been so uncertain, that it has led[5] to a natural distrust of deductions drawn from intensity measurements. Investigators of crystal structures have relied on qualitative methods,[6] since these were in many cases quite adequate. The development of the quantitative analysis has always interested me personally, particularly as a means of attacking the more complicated crystalline structures, and it would seem that at the present time the technique has reached a stage when we can rely on the results. It is my purpose in this paper to attempt a critical survey of the present development of the subject. It is of considerable interest because it is our most direct way of analysing atomic and molecular structure.

In any X-ray examination of a crystalline body, what we actually measure is a series of samples[7] of the coherent radiation scattered in certain definite directions by the unit of the structure. This unit is, in general, the element of pattern of the crystal, while in certain simple cases it may be a single atom.

In the examination of a small body by the microscope, the objective receives the radiation scattered in different directions by the body, and the information about its structure, which we get by viewing the final image, is contained at an earlier stage[8] in these scattered beams. Though the two cases of microscopic and X-ray examination are so similar, there are certain important differences. The scattered beams in the microscope can be combined again to form an image, and in the formation of the image the phase relationship between beams scattered in different directions plays an essential part. In the X-ray problem, since we can only measure the intensity of scattering in each direction, this phase relationship cannot be determined experimentally, though in many cases it can be inferred.[9] Further, the microscope receives the scattered beams over a continuous range of directions, whereas the geometry of the crystalline structure limits our examination to certain directions of scattering. Thus we cannot form directly an image of the crystalline unit which is being illuminated by X-rays. We can only measure experimentally the strength of the scattered beams, and then build up an image piece by piece from the information we have obtained.

It is important to note that in the case of X-ray examination all work is being carried out at what is very nearly the theoretical limit of the resolving power of our instruments. The range of wavelength which it is convenient to use lies between 0.6 Å and 1.5 Å. This range is of sufficiently small wavelength for work with the details of crystal structure, which is always on a scale of several Ångström units, but the wavelengths are inconveniently great for an examination into atomic structure. It is unfortunate from a practical point of view that there is no convenient steady source of radiation between the K lines of the metal palladium, and the very much shorter K lines of tungsten. This difficulty will no doubt be overcome, and a technique of 'ultraviolet' X-ray microscopy will be developed, but at present all the accurate work on intensity of reflection has been done with wavelengths in the neighbourhood of 0.7 Å.

We may conveniently[10] divide the process of analysis into three stages.

a) The experimental measurement of the intensities of the diffracted beams.

b) The reduction of these observations, with the aid of theoretical formulae, to measurements of the amplitudes of the waves scattered by a single unit of the structure, when a wave train of given amplitude falls on it.

c) The building up of the image, or deduction of the form of the unit, from these measurements of scattering in different directions.

2. — History of the use of quantitative methods

The fundamental principles of a mathematical analysis of X-ray reflection were given in Laue's original paper [1], but the precise treatment of intensity of reflection may be said to have been initiated by Darwin [2] with two papers in the *Philosophical Magazine* early in 1914, in which he laid down the basis for a complete theory of X-ray reflection based on the classical laws of electrodynamics.[11] The very fundamental and independent treatment of the whole problem by Ewald [3], along quite different lines, has confirmed Darwin's conclusions in all essentials. These papers established the following important points.

1. Two formulae for the intensity of X-ray reflection can be deduced, depending on the assumptions which are made. The first of these has since come to be known as the formula for the 'ideally imperfect crystal' or 'mosaic crystal'.[a] It holds for a crystal in which the homogeneous blocks are so small that the reduction in intensity of a ray passing through each block, and being partly reflected by it, is wholly accounted for by the ordinary absorption coefficient. This case is simple to treat from a mathematical point of view, and in actual fact many crystals approach this physical condition of a perfect mosaic.

The second formula applies to reflection by an ideally perfect crystal. Here ordinary[12] absorption plays no part in intensity of reflection. This is perfect over a finite range of glancing angles, all radiation being reflected within this range. The range depends on the efficiency of the atom planes in scattering. The second formula is entirely different from the first, and leads to numerical results of a different order of magnitude.

2. The actual intensity of reflection in the case of rocksalt is of the order to be expected from the imperfect crystal formula.

3. The observed rapid decline in intensity of the high orders is only partly accounted for by the formula for reflection, and must be due in addition to the spatial distribution of scattering matter in the atoms (electron distribution).

4. When a crystal is so perfect that it is necessary to allow for the interaction of the separate planes, the transmitted beam is extinguished more rapidly than corresponds to the true absorption of the crystal (extinction).

5. There exists a refractive index for both crystalline and amorphous substances, slightly less than unity, which causes small deviations from the law $n\lambda = 2d\sin\theta$.

Another important factor in intensity of reflection had been already examined theoretically by Debye [4], this being the diminution in

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[a] I believe we owe to Ewald the happy suggestion of the word 'mosaic'.

intensity with rising temperature due to atomic movement. Though subsequent work has put Debye's and Darwin's formulae in modified and more convenient forms, the essential features were all contained in these early papers.

On the experimental side,the first accurate quantitative measurements were made by W. H. Bragg [5].[13] The crystal was moved with constant angular velocity through the reflecting position, and the total amount of reflected radiation measured. He showed that the reflection[14] from rocksalt for a series of faces lay on a smooth curve when plotted against the sine of the glancing angle, emphasising that a definite physical constant was being measured. This method of measurement has since been widely used. The quantity $\frac{E\omega}{I}$, where E is the total energy of radiation[15] reflected, ω the angular velocity of rotation, and I the total radiation falling on the crystal face per second, is independent of the experimental arrangements, and is a constant for a given reflection from a mosaic crystal; it is generally termed the '*integrated reflection*'.[16] It is related in a simple way to the energy measurements from a powdered crystal, which have also been employed for accurate quantitative work. W. H. Bragg's original measurements were comparisons[17] of this quantity for different faces, not absolute measurements in which the strength of an incident beam was considered.

W. H. Bragg further demonstrated the existence of the extinction effect predicted by Darwin, by passing X-rays through a diamond crystal set for reflection and obtaining an increased absorption. He made measurements of the diminution in intensity of reflection[18] with rising temperature predicted by Debye, and observed[19] by Laue, and showed that the effect was of the expected order. In the Bakerian Lecture in 1915 [6] he described measurements in the intensity of a very perfect crystal, calcite, which seemed to show that the intensity was proportional to the scattering power of the atomic planes and not to the square of the power (this is to be expected from the formula for reflection by a perfect crystal). In the same address he proposed the use of the Fourier method of interpreting the measurements[20] which has been recently used with such success by Duane, Havighurst, and Compton, and which is dealt with in the fourth section of this summary.[21] At about the same time, Debye and Compton independently discussed the influence of electronic distribution in the atom on the intensity of reflection.

The next step was made by Compton [7] in 1917. Darwin's formula for the mosaic crystal was deduced by a different method, and was applied to the interpretation of W. H. Bragg's results with rocksalt. Compton concluded that the electronic distribution in the atoms was of the type to be expected from Bohr's atomic model. Compton then published the first measurements of the *absolute* intensity of ref1exion. A monochromatic beam of X-rays was obtained by reflection from a crystal, and this was reflected by a second rotating crystal (rocksalt and[22] calcite). The absolute value of the integrated reflection $\frac{E\omega}{I}$ was found to be of the right order for rocksalt when calculated by the imperfect crystal formula, but to be very low for calcite indicating strong extinction or a wrong formula, in the second case.

In 1921 and 1922 I published with James and Bosanquet a series of measurements on rocksalt in which we tried to obtain a high accuracy. We made absolute measurements of intensity for the strongest reflections,[a] and compared the weaker reflections with them. Our main contributions in these papers were a more accurate set of measurements of integrated reflection for a large number of planes, and a method for estimating and correcting for the effect of extinction. As Darwin showed in a paper in 1922 [9] on the theoretical interpretation of our results, we only succeeded in correcting for extinction of the kind he termed 'secondary' and not for 'primary' extinction.[b] Since then measurements by Havighurst [10], by Harris, Bates and McInnes[23] [11] and by Bearden [12] have been made on the reflecting power of powdered sodium chloride when extinction is absent. Their measurements have agreed with ours very closely indeed, confirming one's faith in intensity measurements, and showing that we were fortunate in choosing a crystal for our examination where primary extinction was very small. In the same papers we tried to make a careful analysis of the results in order to find how much information about atomic structure could be legitimately deduced from them, and we published curves showing the electron distribution in sodium and chlorine[24] atoms.

In this discussion, I have refrained from any reference to the question of reflection by 'perfect' crystals. The formula for reflection by such crystals was first obtained by Darwin, and has been arrived at independently by Ewald. The reflection by such crystals has been examined amongst others by Bergen Davis[25] and Stempel [13], and by Mark [14] and predictions of the theory have been verified. It is not considered

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[a] In our paper we failed to give due acknowledgement to Compton's absolute measurements in 1917 of which we were not aware at the time.

[b] Primary extinction is an excessive absorption of the beam which is being reflected in each homogeneous block of crystal, secondary extinction a statistical excessive absorption of the beam in the many small blocks of a mosaic crystal.

here, because I wish to confine the discussion to those cases where a comparison of the intensity of incident and reflected radiation leads to accurate quantitative estimates of the distribution of scattering matter. This ideal can be attained with actual crystals,[26] when they are of the imperfect or mosaic type, though allowance for extinction is sometimes difficult in the case of the stronger reflections. On the other hand, it is far more difficult to know what one is measuring in the case of crystals which approximate to the perfect type. It is a fortunate circumstance that mathematical formulae can be applied most easily to the type of imperfect crystal more common in nature.

3. — RESULTS OF QUANTITATIVE ANALYSIS

For the sake of conciseness, only one of the many intensity formulae will be given here, for it illustrates the essential features of them all. Let us suppose that the integrated reflection is being measured when X-rays fall on the face of a rotating crystal of the mosaic type. We then have

$$\rho = \frac{Q}{2\mu}\frac{1+\cos^2 2\theta}{2} \; .$$

(a) μ is the effective absorption coefficient, which may be greater than the normal coefficient, owing to the existence of extinction at the reflecting angle.

(b) The factor $\frac{1+\cos^2 2\theta}{2}$ is the ‘polarisation factor’, which arises because the incident rays are assumed to be unpolarised.

(c)

$$Q = \left(\frac{Ne^2}{mc^2}F\right)^2 \frac{\lambda^3}{\sin 2\theta} \; ,$$

where e and m are the electronic constants,[27] c the velocity of light, λ the wavelength used, N the number of scattering units per unit volume, and θ the glancing angle.

(d) F is the quantity we are seeking to deduce. It represents the scattering power of the crystal unit in the direction under consideration, measured in terms of the scattering power of a single[28] electron according to the classical formula of J. J. Thomson. It is defined by Ewald as the ‘Atomfaktor’[29] when it applies to a single atom.

Formulae applicable to other experimental arrangements (the powder method for instance) are very similar, and contain the same quantity Q. Our measurements of reflection thus lead to values of Q, and so of F, since all other quantities in the formulae are known. Measurements on a given crystal yield a series of values for F, and all the information that can be found out about this crystalline or atomic structure is represented by these values. They are the same for the same crystal whatever wavelength is employed (since F is a function of $\frac{\sin\theta}{\lambda}$), though of course with shorter wavelength we have the advantage of measuring a much greater number of these coefficients (increased resolving power).

At this stage the effect of the thermal agitation of the atom will be considered as influencing the value of F. If we wish to make deductions about atomic structure, the thermal agitation must be taken into account. Allowance for it is a complicated matter, because not only do some atoms move more than others, but also they change their relative mean positions as the temperature alters in the more complex crystals.

This will be dealt with more fully below.

A series of examples will now be given to show that these quantitative formulae, when tested, lead to results which indicate that the theory is on the right lines. It is perhaps more convincing to study the results obtained with very simple crystals, though I think that the success of the theory in analysing highly complex structures is also very strong evidence, because we have covered such a wide range of substances.

In the simple crystals, where the positions of the atoms are definite, we can get the scattering power of individual atoms. The results should both indicate the correct number of electrons in the atom, and should outline an atom of about the right size. When F is plotted against $\frac{\sin\theta}{\lambda}$ its value should tend to the number, N, of electrons in the atom for small values of $\frac{\sin\theta}{\lambda}$, and should fall away as $\frac{\sin\theta}{\lambda}$ increases, at a rate which is reasonably explained by the spatial extension of the atom. In Fig. 1, the full lines give F curves obtained experimentally by various observers. The dotted lines are F curves calculated for the generalised atomic model of Thomas [15], of appropriate atomic number. The Thomas atomic model, which has been shown for comparison, is most useful as it gives us the approximate electronic distribution in an atom of any atomic number. Thomas calculates an ideal distribution of electrons in an atom of high atomic number. He assumes spherical symmetry for the atom, and supposes that ‘electrons are distributed uniformly in the six-dimensional phase-space for the motion of an electron, at the rate

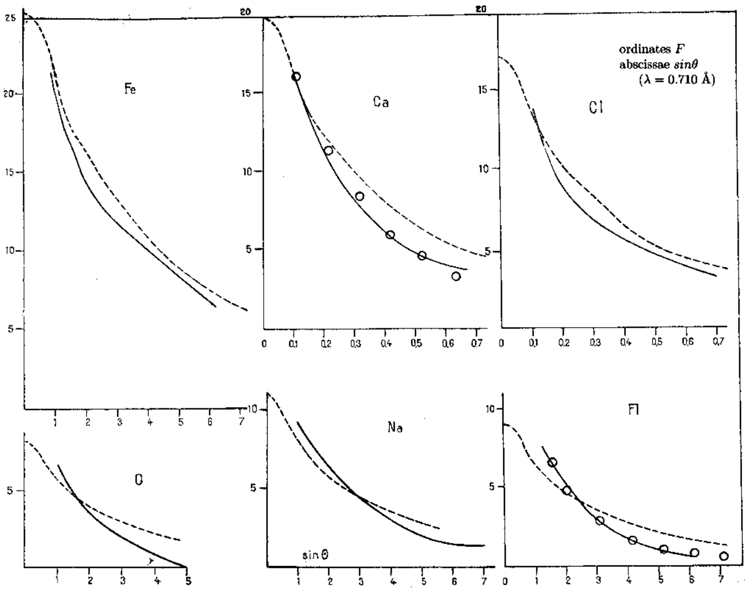

Fig. 1.

of two for each h^3 of (six) volume'.[30] He thus obtains an ideal electron atmosphere around the nucleus, the constants of which can be simply adjusted[31] so as to be suitable for any given nuclear charge. It is of course to be expected that the lower the atomic weight, the more the actual distribution of scattering matter will depart from this arrangement, and will reflect the idiosyncrasies of the particular atom in question. The figure will show, however, that the actual curves are very similar to those calculated for Thomas' models. In particular, it will be clear that they tend to maximum values not far removed from the number of electrons in the atom in each case. The general agreement between the observed and calculated F curves must mean that our measurements of F are outlining a picture of the atom. The agreement holds also for other atomic models than those of Thomas, which all lead to atoms with approximately the same spatial extension and electronic distribution, as is well known.

All these measurements of F necessitate absolute values for the integrated reflection. It is not necessary to measure these directly in each case. When any one reflection has been measured in absolute value (by comparison of incident and reflected radiation), other crystals may be compared with it. The standard which has been used in every case, as far as I am aware, is the rocksalt crystal. Absolute measurements on this have been made by Compton [7], by Bragg, James and Bosanquet [8], and by Wasastjerna [18] which agree satisfactorily with each other.

4. — INTERPRETATION OF MEASUREMENTS OF F

In interpreting these measurements of scattering power, we may either calculate the scattering of a proposed atomic model and compare it with the observed F curve, or we may use the observations to calculate the distribution of scattering matter directly. The latter method is the more attractive, and in the hands of Duane, Havighurst, and Compton it has yielded highly interesting 'images' of the atomic structure seen by X-rays. There is a close analogy between the examination of a series of parallel planes by means of X-rays, and the examination of a diffraction grating, by a microscope, which is considered in Abbe's theory of microscopic vision.[a] The objective of the microscope may be considered as receiving a limited number of orders of spectra from the grating. These spectra in their turn build up the image viewed by the eyepiece, and the perfection of this image depends on the number of spectra received. The strength of each spectral order depends on the magnitude of the corresponding coefficient in that Fourier series which represents the amplitude of the light transmitted at each point of the grating. The extension of this well-known optical principle to the X-ray field was suggested by W. H. Bragg [6] in 1915. He had formed the conclusion[32] that the amplitudes of the scattered wave from rocksalt were inversely proportional to the square of the order of reflection, and he showed that[33] 'the periodic function which represents the density of the medium must therefore be of the form[34]

$$\text{const} + \frac{\cos 2\pi\frac{x}{d}}{1^2} + \frac{\cos 4\pi\frac{x}{d}}{2^2} + \dots + \frac{\cos 2n\pi\frac{x}{d}}{n^2} + \dots \text{'}$$

and in this way built up a curve showing the periodic density of the rocksalt grating. The method was not applied, however, to the much more accurate measurements which are now available until recently, when Duane and Havighurst showed how much could be done with it.

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[a] See for instance the discussion of this theory and of A. B. Porter's experiments to illustrate it in Wood's *Optics*, Chapter VIII.

Duane independently arrived at a more general formula of the same type, giving the density of scattering matter at any point in the whole crystal as a triple Fourier series, whose coefficients depend on the intensity of reflection from planes of all possible indices. Havighurst applied this principle to our measurements of rocksalt, and to measurements which he has made on other crystals, and obtained a picture of the relative density of scattering matter along certain lines in these crystals. Compton made the further step of putting the formulae in a form which gives the absolute density of electronic distribution (assuming the scattering to be by electrons obeying the classical laws). Compton gives a very full discussion of the whole matter in his book *X-rays and Electrons*.[35] It is not only an extremely attractive way of making clear just what has been achieved by the X-ray analysis, but also the most direct method of determining the structure.

The formula for the distribution of scattering matter in parallel sheets, for a crystal with a centre of symmetry, is given by Compton as follows

$$P_z = \frac{Z}{a} + \frac{2}{a}\sum_1^\infty F_n \cos\frac{2\pi nz}{a} \ .$$

Here z is measured perpendicularly to the planes which are spaced a distance a apart. $P_z dz$ is the amount of scattering matter between planes at distances z and $z + dz$, and Z $(= \int_0^a P_z dz)$ is the total scattering matter of the crystal unit. This is a simplified form of Duane's formula for a Fourier series of which the general term is

$$A_{n_1n_2n_3} \sin\left(\frac{2\pi n_1 x}{a_1} - \delta_{n_1}\right) \sin\left(\frac{2\pi n_2 y}{a_2} - \delta_{n_2}\right) \sin\left(\frac{2\pi n_3 z}{a_3} - \delta_{n_3}\right) \ ,$$

$A_{n_1n_2n_3}$ being proportional to the amplitude of the scattered wave from the plane $(n_1n_2n_3)$.

Another Fourier series, due to Compton, gives the radial distribution of scattering matter, i.e. the values of U_n where $U_n dr$ is the amount of scattering matter between radii r and $r + dr$

$$U_n = \frac{8\pi r}{a^2}\sum_1^\infty nF_n \sin\frac{2\pi nr}{a} \ ,$$

where a is chosen so that values of F occur at convenient intervals on the graph for F.

If we know the values of F for a given atom over a sufficiently wide range, we can build up an image of the atom either as a 'sheet distribution' parallel to a plane, or as a radial distribution of scattering matter around the nucleus. In using these methods of analysis, however, it is very necessary to remember that we are working right at the limit of resolving power of our instruments, and in fact are attempting a more ambitious problem than in the corresponding optical case. In A. B. Porter's experiments to test Abbe's theory, he viewed the image of a diffraction grating and removed any desired group of diffracted rays by cutting them off with a screen. The first order gives blurred lines, four or five orders give sharper lines with a fine dark line down the centre, eight orders give two dark lines down the centre of each bright line and so forth. These imperfect images are due to the absences of the higher members in building up the Fourier series. In exactly the same way we get false detail in our X-ray image, owing to ignorance of the values of the higher members in the F curve. Similarly, the fine structure which actually exists may be glossed over, since by using a wavelength of 0.7 Å, we cannot hope to 'resolve' details of atomic structure on a scale of less than half this value.

The ignorance of the values of higher members of the Fourier series matters much less in the curve of sheet distribution than in that for radial distribution, since the latter converges far more slowly. Examples of the Fourier method of analysis are given in the next paragraph.

As opposed to this method of building up an image from the X-ray results,[36] we may make an atomic model and test it by calculating an F curve for it which can be compared with that obtained experimentally. This is the most satisfactory method of testing models arrived at by other lines of research, for nothing has to be assumed about the values of the higher coefficients F. It is of course again true that our test only applies to details of the proposed model on a scale comparable with the wavelength we are using. Since we can reflect X-rays right back from an atomic plane, we may get a resolving power for a given wavelength with the X-ray method twice as great as the best the microscope can yield.

It is perhaps worth mentioning the methods I used with James and Bosanquet in our determination of the electronic distribution in sodium and chlorine in 1922. We tried to avoid extrapolations of the F curve beyond the limit of experimental investigation. We divided the atom arbitrarily into a set of shells, with an unknown number of electrons in each shell. These unknowns were evaluated by making the scattering due to them fit the F curve over the observed range, this being simply done by solving simultaneous linear equations. We found we got much the same type of distribution however the shells were chosen, and that a limit to the electronic distribution at a radius of about 1.1 Å in sodium

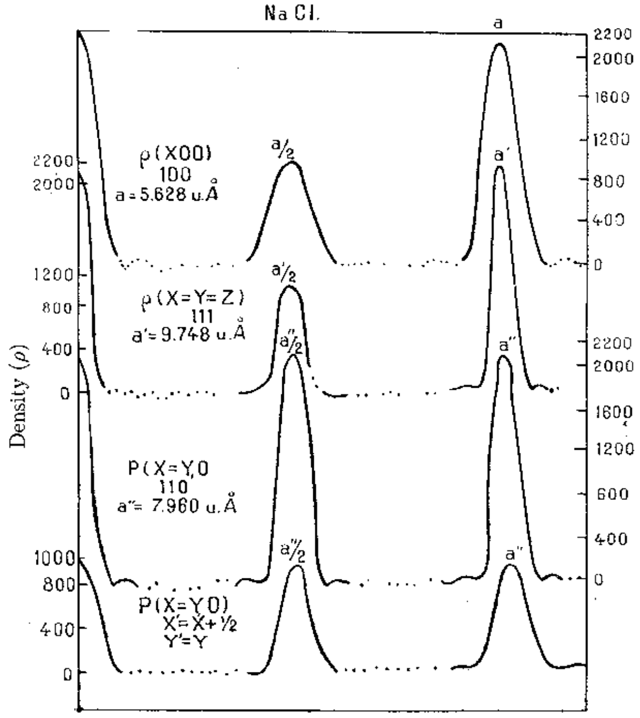

Fig. 2.

and 1.8 Å in chlorine was clearly indicated. Our distribution corresponds in its general outline to that found by the much more direct Fourier analysis, as the examples in paragraph 7 will show.

5. — Examples of analysis

We owe to Duane [20] the appreciation of the very attractive way in which the Fourier analysis represents the results of X-ray examinations. It has the great merit of representing, in the form of a single curve, the information yielded by all orders of reflection from a given plane, or from the whole crystal. It is of course only an alternative way of interpreting the results, and the deductions we can make about atomic or molecular structures depend in the end on the extent to which we can

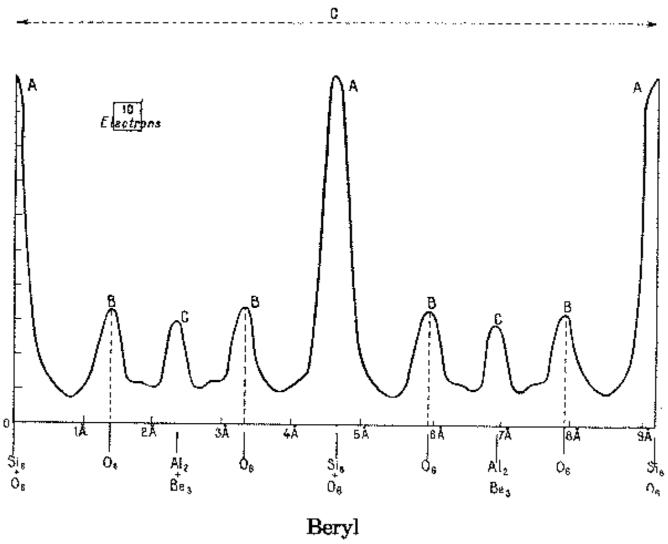

Fig. 3a. — Distribution of electrons in sheets parallel to 0001.[37]

trust our experimental observations, and not on the method of analysis we use. The Fourier method is so direct however, and its significance so easy to grasp, that Duane's introduction of it marks a great advance in technique of analysis.

I have reserved to paragraph 7 the more difficult problem of the arrangement of scattering matter in the atoms themselves, and the examples given here are of a simpler character. They illustrate the application of analysis to the general problem of the distribution of scattering matter in the whole crystal, when we are not so near the limit of resolving power. The curves in Fig. 2 represent the first application of the new method of Fourier analysis to accurate data, carried out by Havighurst [21] in 1925. He used our determinations of F for sodium chloride, and Duane's three-dimensional Fourier series, and calculated the density of scattering matter along a cube edge through sodium and chlorine centres, along a cube diagonal through the same atoms, and along two face diagonals chosen so as to pass through chlorine atoms alone or sodium atoms alone in the crystal. The atoms show as peaks in the density distribution. In the other examples, the formula for distribution in sheets has been

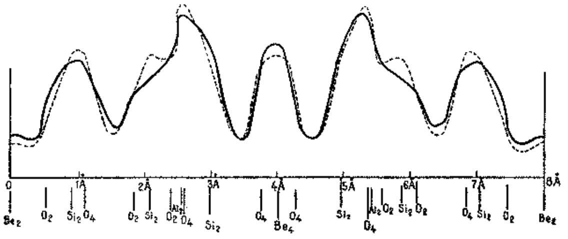

Fig. 3b. — Distribution of electrons in sheets parallel to $10\bar{1}0$.[39]

applied to some results we have obtained in our work on crystal structure at Manchester. I have given them because I feel they are convincing evidence of the power of quantitative measurements, and show that all methods of interpretations lead to the same results.

Mr West and I [22] recently analysed the hexagonal crystal beryl, $Be_3Al_2Si_6O_{18}$,[38] which has a structure of some complexity, depending on seven parameters. We obtained the atomic positions by the usual method of analysis, using more or less known F curves for the atoms in the crystal, and moving them about till we explained the observed Fs due to the crystal unit. Fig. 3 shows the reinterpretations of this result by the Fourier method. Fig. 3a gives the electron density in sheets perpendicular to the principal axis of the crystal, which is of a very simple type. The particular point to be noted is the correspondence between the position of the line B in the figure and the hump of the Fourier analysis. The line B marks the position of a group of oxygen atoms which lies between two other groups A and C fixed by symmetry, the position of B being fixed by a parameter found by familiar methods of crystal analysis. The hump represents the same group fixed by the Fourier analysis, and it will be seen how closely they correspond. In Figs. 3b and 3c more complex sets of planes are shown. The dotted curve represents the interpretation of our results by Fourier analysis. The full curve is got by adding together the humps due to the separate atoms shown below, the position of these having been obtained by our X-ray analysis and their sizes by the aid of the curve in Fig. 3a in which the contribution of the atoms can be separated out. The correspondence between the two shows that the older methods and the Fourier analysis agree. It is to be noted that the crystal

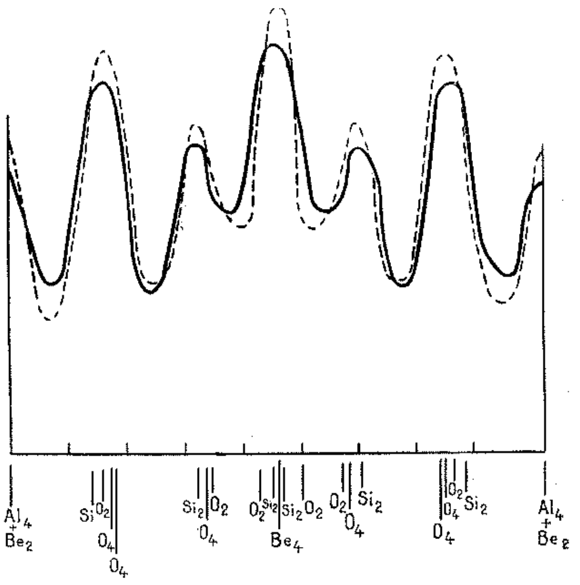

Fig. 3c. — Distribution of electrons in sheets parallel to $11\bar{2}0$.[42]

had first to be analysed by the older methods, in order that the sizes of the Fourier coefficients might be known.

In Fig. 4 I have given a set of curves for the alums, recently analysed by Professor Cork [23]. The alums are complicated cubic crystals with such formulae as $KAl(SO_4)_2.12\,H_2O$. Wyckoff[40] has shown that the potassium and aluminium atoms[41] occupy the same positions in the cubic cell as the sodium atom in rocksalt. Now we can replace the potassium by ammonium, rubidium, caesium, or thallium, and the aluminium by chromium, or other trivalent metals. Though the positions of the other atoms in the crystals are not yet known, they will presumably be much the same in all these crystals. If we represent by a Fourier series the quantitative measurements of the alums, we would expect the density of scattering matter to vary from crystal to crystal at the points occupied by the metal atoms, but to remain constant elsewhere. The curves show this in the most interesting[43] way.

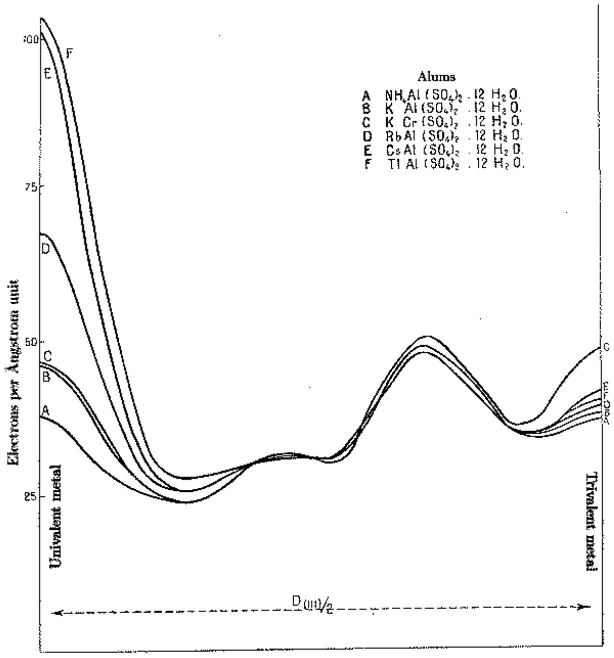

Fig. 4.

The effect of heat motion on the movements of the atoms has already been mentioned. It was first treated theoretically by Debye [4]. Recently Waller [24] has recalculated Debye's formula, and has arrived at a modified form of it. Debye found that the intensities of the interference maxima in a simple crystal should be multiplied by a factor e^{-M}, where

$$M = \frac{6h^2}{\mu k \Theta} \frac{\varphi(x)}{x} \frac{\sin^2 \theta}{\lambda^2} \ ,$$

$$x = \frac{\Theta}{T} = \frac{\text{characteristic temperature of crystal}}{\text{absolute temperature}} \ .$$

Without going into further detail, it is sufficient to note that Waller's formula differs from Debye's by making the factor e^{-2M}, not e^{-M}. James and Miss Firth [25] have recently carried out a series of measurements

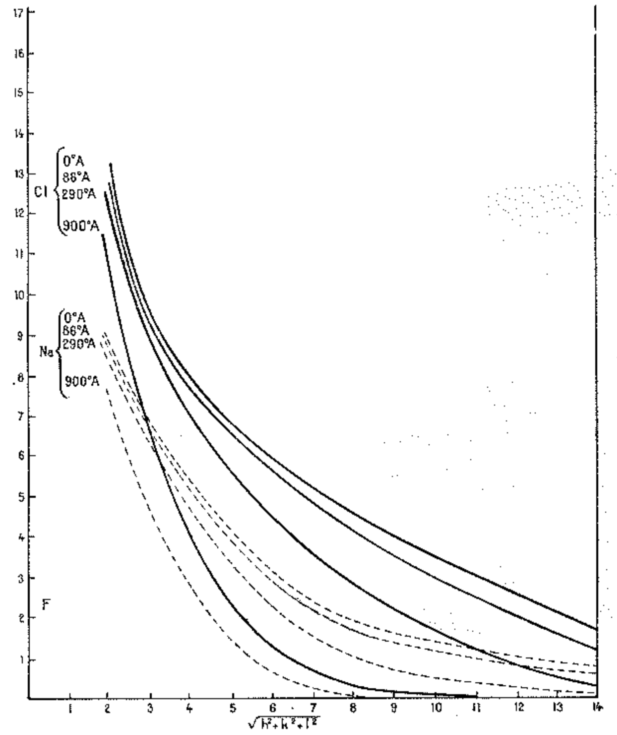

Fig. 5.

for rocksalt between the temperatures 86° abs. and 900° abs. They find that Waller's formula is very closely followed up to 500° abs., though at higher temperatures the decline in intensity is even more rapid, as is perhaps to be expected owing to the crystal becoming more loosely bound. I have given the results of the measurements in Figs. 5 and 6, both as an example of the type of information which can be got from X-ray measurements, and because these actual figures are of interest as a set of careful and accurate measurements of scattering power.

Fig. 5 shows the F curves for sodium and chlorine at different temperatures. The rapid decline in intensity for the higher orders will be realised when it is remembered that they are proportional to F^2. The

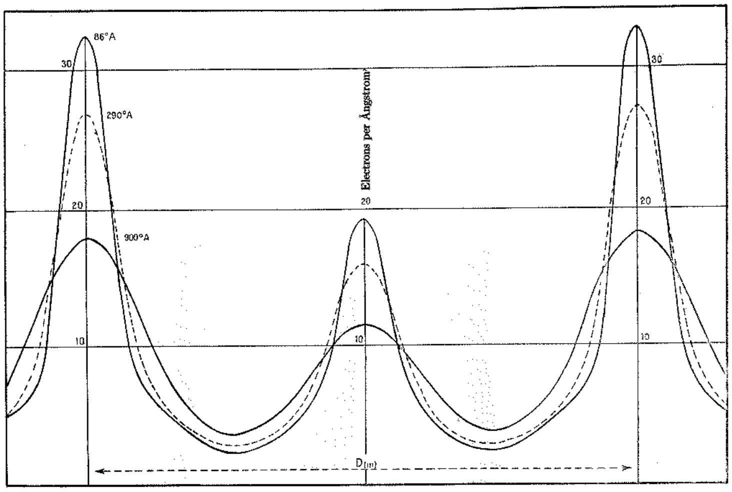

Fig. 6.

curve for absolute zero is an extrapolation from the others, following the Debye formula as modified by Waller.

In Fig. 6 the same results are interpreted by the Fourier analysis. The curve at room temperatures for NaCl is practically identical with the interpretation of our earlier figures by Compton, in his book *X-rays and Electrons*,[44] though the figures on which it is based should be more accurate.[45] The curves show the manner in which the sharply defined peaks due to Cl and Na at low temperatures become diffuse owing to heat motion at the higher temperatures.

Several interesting points arise in connection with this analysis. In the first place, James and Firth[46] find that the heat factor is different for sodium and chlorine, the sodium atoms moving with greater average amplitudes than the chlorine atoms. This has a very interesting bearing on the crystal dynamics which is being further investigated by Waller. To a first approximation both atoms are affected equally by the elastic waves travelling through the crystal, but in a further approximation it can be seen that the sodium atoms are more loosely bound than the

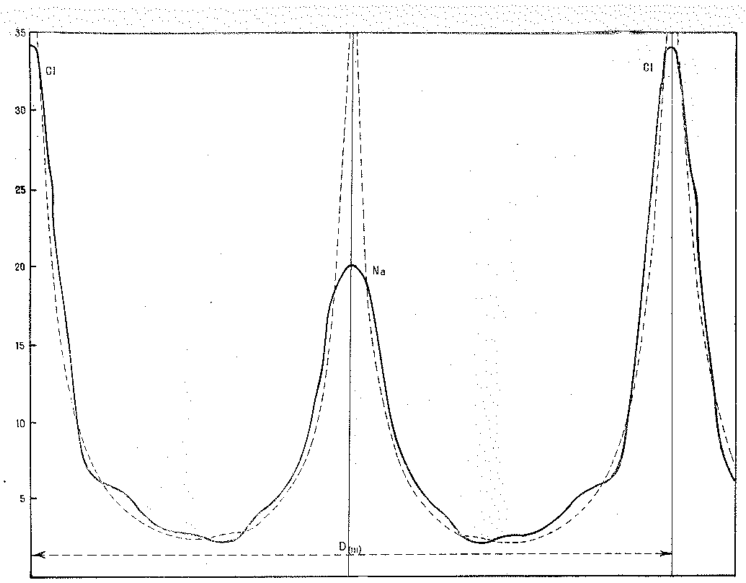

Fig. 7.

chlorine atoms. If an atom of either kind were only fixed in position by the six atoms immediately surrounding it, Waller has shown that there would be no difference between the motions of a sodium atom between six chlorine atoms, or a chlorine atom between six sodium atoms. However, the chlorine is more firmly pinned in position because it has in addition twelve large chlorine neighbours, whereas the sodium atom is much less influenced by the twelve nearest sodium atoms. Hence arises the difference in their heat motions. It is important to find the correct method for reducing observations to absolute zero, and this difference in heat motion must be satisfactorily analysed before this is possible.

In the second place, the accuracy which can be attained by the experimental measurements holds out some hope that we may be able to test directly whether there is zero-point energy[47] or not. This is being investigated by James and Waller. If a reliable atomic model is available, it would seem that the measurements can tell whether there is vibration at absolute zero or not, for the theoretical diminution in intensity due to the vibration is much larger than the experimental error in measuring

F. I feel considerable diffidence in speaking of the question of zero-point energy, and would like to have the advice of the mathematical physicists present.

We may calculate, either from the measured heat factor or directly from the Fourier analyses, the average amplitude of vibration for different temperatures. James and Firth find by both methods, for instance, that at room temperature the mean amplitude of vibration for both atoms is 0.21 Å, and at 900° abs. it is about 0.58 Å. They examined the form which the Fourier curve at 0° abs. assumes when it is deformed by supposing all the atoms to be in vibration with the same mean amplitude.

It has been already remarked that the observed F curves for atoms are very similar to those calculated for the Thomas atomic model. The same comparison may be made between the distributions of scattering matter. In Fig. 7 the distribution in sheets for NaCl at absolute zero is shown as a full curve. The dotted curve shows the horizontal distribution in sheets for atoms of atomic number 17 and 11. In Thomas' model the density rises towards an infinite value very close to the nucleus, and this is represented by the very sharp peaks at the atomic centres in the dotted curve. We would not expect the observed distribution to correspond to the actual Thomas distribution at these points. Throughout the rest of the crystal the distribution is very similar. The comparison is interesting, because it shows how delicate a matter it is to get the fine detail of atomic structure from the observations. Thomas' distribution is quite continuous and takes no account of K, L and M sets of electrons. The slight departures of the observed curve from the smooth Thomas curve represent the experimental evidence for the existence of all the individual features of the atom.

6. — The mechanism of X-ray scattering

Before going on to discuss the application of the analysis to atomic structure, it is necessary to consider what is being measured when a distribution of scattering matter is deduced from the X-ray results. The classical treatment regards the atom as containing a number of electrons, each of which scatters radiation according to the formula of J. J. Thomson. Since a vast number of atoms contribute to the reflection by a single crystal plane, we should obtain a picture of the *average electronic distribution.* The quantity F should thus tend to a maximum value, at small angles of scattering, equal to the number of electrons in the atom, and should fall away owing to their spatial distribution as $\frac{\sin\theta}{\lambda}$ increases. The observed[48] F curves are of this character, as has been seen. When interpreted as an atomic distribution, they give atoms containing the correct number of electrons, and this seems satisfactory from the classical viewpoint. On the other hand, the evidence of the Compton effect would appear at first sight to cast doubt on the whole of our analysis. What we are measuring is essentially the *coherent* radiation diffracted by the crystal, whereas the Compton effect shows that a part of the radiation which is scattered is of different wavelength. Further, this radiation of different wavelength is included with the coherent radiation, when the total amount of scattered radiation is measured, and found to agree under suitable conditions with the amount predicted by J. J. Thomson's formula. It would therefore seem wrong to assume that we obtain a true picture of electronic distribution by the aid of measurements on the coherent radiation alone.

Even before the advent of the new mechanics, Compton's original treatment of the effect which he discovered suggested a way out of this difficulty. The recoil electron is given an amount of energy

$$2\frac{h^2}{m}\frac{\nu'}{\nu}\left(\frac{\sin\theta}{\lambda}\right)^2 ,$$

where ν and ν' are the frequencies of the modified and unmodified radiations. If the electron is ejected from the atom the radiation is modified in wavelength, if not coherent waves are scattered. Since there is little modified scattering at small angles, the F curve will tend to a maximum equal to the number of electrons in the atom, and any interpretation of the curve will give an atom containing the correct number of electrons. As $\frac{\sin\theta}{\lambda}$ increases, more and more of the scattered radiation will be modified, and in calculating the F curve this must be taken into account. However, if $\frac{\nu'}{\nu}$ is not far from unity, the F curve will remain a function of $\frac{\sin\theta}{\lambda}$, since whatever criterion is applied for the scattering of modified or unmodified radiation, it will depend on the energy imparted to the scattering electron, which is itself a function of $\frac{\sin\theta}{\lambda}$. Our X-ray analysis would thus give us an untrue picture of the atom, but one which is consistently the same whatever wavelength is employed. Williams [27] and Jauncey [28] have recalculated F curves from atomic models using this criterion, and found a better fit to the experimental curves when the Compton effect was taken into account. (Examples of this closer approximation will be found in the paper by

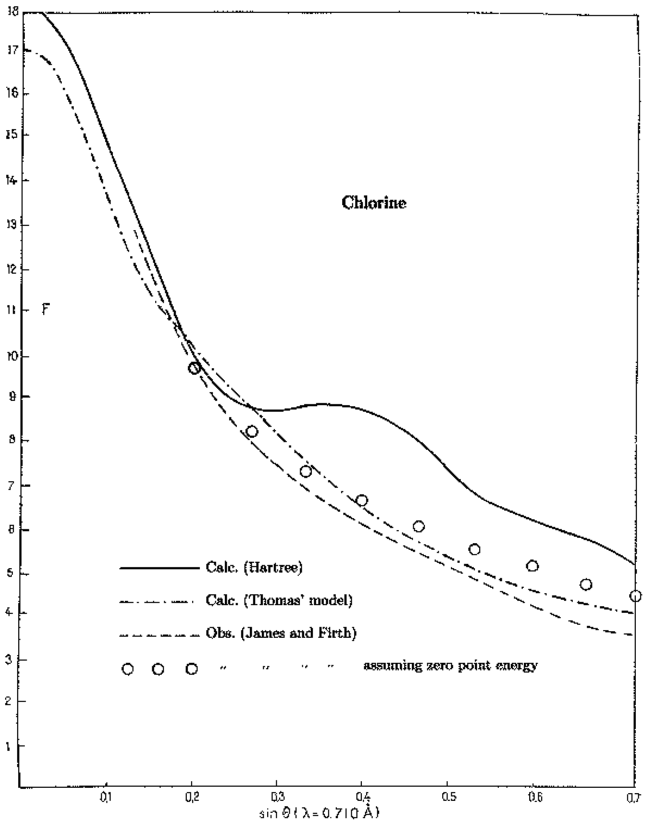

Fig. 8.

Williams [27] in 1926. See also a discussion by Kallmann and Mark [26]).[49]

The point at issue is illustrated by the curves in Fig. 8. Three F curves for chlorine are plotted in the figure. The dotted line represents the observed F curve (James and Firth). The continuous line is the F curve calculated from Hartree's [29] atomic model for chlorine. It shows a hump at a value of $\sin\theta$ of 0.4, which is not present in the observed curve. This hump arises from the fact that the outer electrons in the chlorine model give negative values for F just short of this point,[50] and positive values again at the point itself. All atomic models calculated with electronic orbits show similar irregularities which are not actually observed. When, however, the Compton effect is taken into consideration, these outer electrons are found to give a very small contribution to the F curve at the large angles where the humps[51] occur, because they scatter so much modified radiation. The allowance for the Compton effect smooths out the hump, and leads to F curves much more like those observed. The third curve shows the F curve due to the continuous Thomas distribution and is a close fit to the observed curve.

I have quoted from a note by Dr Ivar Waller, in the following tentative summary of the interpretation which the new mechanics gives us of this phenomenon.[a] In a recent letter to *Nature* [30], Waller discusses the transition for the whole range from ordinary dispersion into Compton effect. His note only refers to scattering by a single electron, but it can probably be extended to many-electron atoms. Waves of continually decreasing wavelength are supposed to fall upon the atom, and the transition is traced through the following stages.

a) While the wavelength of the radiation remains long compared with atomic dimensions, the dispersion formula for optical frequencies gradually transforms into the scattering for free electrons given by the classical J. J. Thomson formula. This formula holds approximately to[52] wavelengths approaching atomic dimensions.

b) At this point the scattering of coherent radiation will diminish, owing to interference, and become more concentrated in the forward direction of the incident light. This is the phenomenon we are studying, with X-rays, and our F curves map out the distribution of the coherent radiation where the wavelength is of atomic dimensions.

c) At the same time, the scattering of incoherent radiation will become appreciable, and approximate more and more closely in change of wavelength and intensity distribution to the Compton effect. It will have practically merged into the Compton effect when the momentum of a quantum of the incident light is large compared with that corresponding to electronic motions in the atom.

d) Up to this point the Thomson formula holds for the total intensity

[a] Space forbids a reference to the many theoretical papers which have contributed towards this interpretation.

of light scattered in any direction, coherent and incoherent radiation being summed together. It first ceases to hold, when the frequency displacement due to the Compton effect is no longer small compared with the frequency of the incident light.

The point of importance for our present problem is that 'the coherent part of the radiation is to be directly calculated from that continuous distribution of electricity which is defined by the Schrödinger density-distribution in the initial state of the atom'. The classical treatment supposes each point electron to scatter according to the J. J. Thomson formula in all directions. In the new treatment, the electron is replaced by a spatial distribution of scattering matter, and so each electron has an 'F curve' of its own. It will still scatter coherent radiation in all directions, but its amount will fall away from that given by the classical formula owing to interference as $\frac{\sin\theta}{\lambda}$ increases, and this decline will be much more rapid for the more diffuse outer electrons than for the concentrated inner electrons. The total amount of radiation T scattered in any direction by the electron is given by the Thomson formula. A fraction f^2T will be coherent, and will be calculated by the laws of interference from the Schrödinger distribution, and the remainder, $(1-f^2)T$, will be incoherent. Thus the total coherent radiation will be F^2T where F is calculated from the Schrödinger distribution for the whole atom. An amount $(N-\sum f^2)T$ will be scattered with change of wavelength. Our measurements of X-ray diffraction, if this be true, can be trusted to measure the Schrödinger continuous distribution of electricity in the crystal lattice.

A very interesting point arises in the case where characteristic absorption frequencies of the scattering atom are of shorter wavelength than the radiation which is being scattered. In general, this has not been so when careful intensity measurements have been made since atoms of low atomic weight have alone been investigated. On the classical analogy, we would expect a reversal in phase of the scattered radiation, when an electron has a characteristic frequency greater than that of the incident light. A fascinating[53] experiment by Mark and Szilard [31] has shown that something very like this takes place. They investigated the (111) and (333) reflections of RbBr, which are extremely weak because Rb and Br oppose each other and are nearly equal in atomic number. They found that these 'forbidden' reflections were indeed absent when the soft $\mathrm{Cu_K}$ or hard $\mathrm{Ba_K}$ radiation was used, but that $\mathrm{Sr_K}$ radiation was appreciably reflected ($\mathrm{Sr_{K\alpha}}$ λ 0.871 Å;[54] absorption edges of $\mathrm{Rb_K}$ and $\mathrm{Br_K}$, 0.814 Å and 0.918 Å). The atoms are differentiated because a reversal of phase in scattering by the K electrons takes place in the one case and not in the other.

7. — The analysis of atomic structure by X-ray intensity measurements

It has been seen that the intensity measurements assign the correct number of electrons to each atom in a crystal, and indicate a spatial extension of the atoms of the right order. In attempting to make the further step of deducing the arrangement of the electrons in the atom, the limitations of the method begin to be very apparent.

In all cases where analysis has been attempted, the atom has been treated as spherically symmetrical. The analysis is used to determine the amount of scattering matter $U_n dr$ between radii r and $r + dr$. All methods of analysis give a distribution of the same general type. I have given, for instance, a series of analyses of sodium and chlorine in Fig. 9. In these figures, U_n is plotted as ordinate against r as abscissa. The total area of the curve in each case is equal to the number of electrons in each atom, since $\int_0^\infty U_n dr = N$. The full-line curves are our original interpretations of the distribution in sodium and chlorine, based on our 1921 figures.[55] The other curves are the interpretations of the same or closely similar sets of figures[57] by Havighurst [32] and by Compton (*X-rays and Electrons*) using the Fourier method of analysis.

In Fig. 9a are included our analysis of sodium in NaCl, two analyses by Havighurst of sodium in NaCl and NaF obtained by using Duane's triple Fourier series, and an analysis of our figures[58] by Compton using the Fourier formula for radial distribution. It will be seen that the general distribution of scattering matter and the limits of the atom are approximately the same in each case. The same holds for the chlorine curves in Fig. 9b.

The interesting point which is raised is the reality of the humps which are shown by the Fourier analysis. We obtained similar humps in our analysis by means of shells but doubted their reality because we found that if we smoothed them out and recalculated the F curve, it agreed with the observed curve within the limits of experimental error. The technique of measurement has greatly improved since then, and it would even appear from later results that we over-estimated the possible errors of our first determinations of F. It is obvious, however, that great care must still be taken in basing conclusions on the finer details shown by any

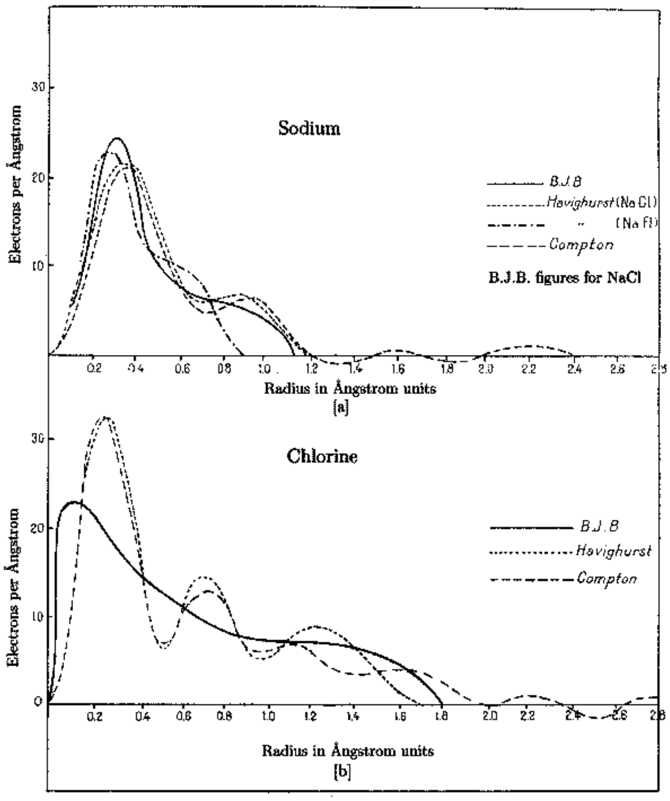

Fig. 9.[56]

method of analysis. The formula which is used in the Fourier analysis,

$$U_n = \frac{4\pi r}{a} \sum_{1}^{\infty} \frac{2nF_n}{a} \sin \frac{2\pi n r}{a} \ ,$$

is one which converges very slowly, since the successive coefficients F_n are multiplied by n. The observed F curve must be extrapolated to a point when F is supposed to fall to zero, and the precise form of the curve reacts very sensitively to the way in which this extrapolation is carried out.

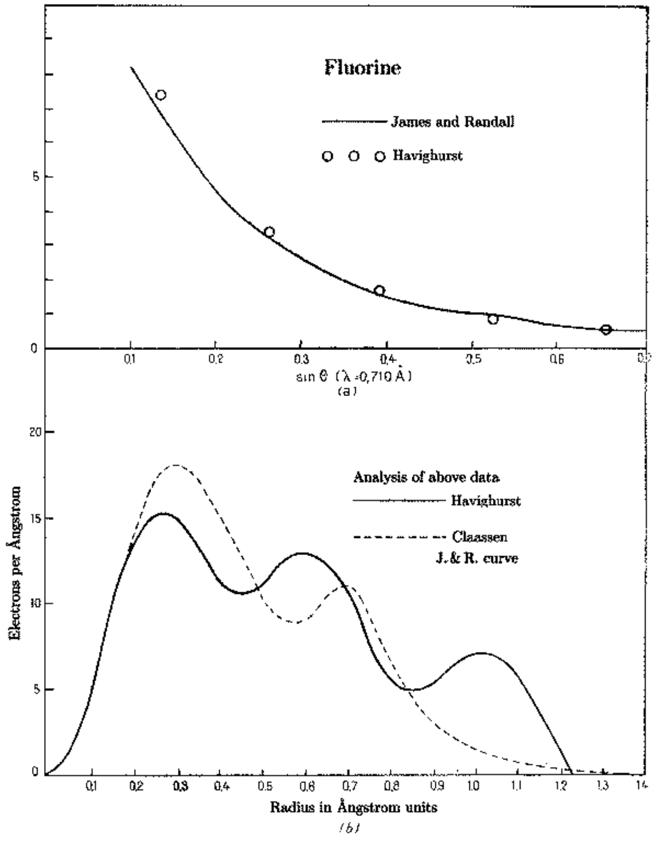

Fig. 10.

The curves in Fig. 10 will illustrate the extent to which the analysis can be considered to give us information about the actual atomic distribution. In Fig. 10a the curve shows the F values for fluorine obtained by James and Randall [17]. The circles are points obtained[59] by Havighurst from measurements on CaF, LiF, NaF;[60] it will be seen that the two sets of experimental data are in very satisfactory agreement. In Fig. 10b I have shown on the one hand Havighurst's interpretations of the F curve drawn through his points, and on the other an analysis carried out by

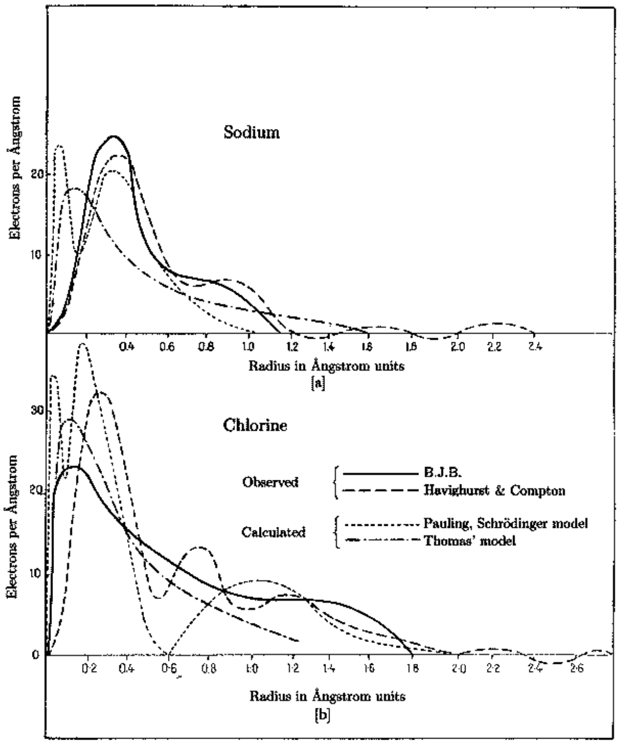

Fig. 11.[62]

Claassen [16] of James and Randall's using the Fourier method. The distributions are the same in their main outlines, but the peaks occur in quite different places.

Compton (*X-rays and Electrons*, p. 167) in discussing his diagrams of radial distribution has remarked that slight differences in the F curves lead to wide differences in details of the curves, and that too much confidence should not be placed on these details. Havighurst [32] discusses the significance of the analysis very fully in his paper on electron distribution in the atoms. Our data are not yet sufficiently accurate or extensive. Nevertheless, we are so near to attaining an accuracy of a satisfactory order, and the results of the analysis seem to indicate so clearly its fundamental correctness, that it appears to be well worth while to pursue enquiry further. Work with shorter wavelengths, and at low temperatures, when heat motion is small and a large range of F values can be measured, should yield us accurate pictures of the atomic structure itself. Given accurate data,[61] the Fourier method of analysis provides a direct way of utilising them.

The radial distribution of scattering power outlined in this way is in general agreement with any reasonable atomic model. We have seen, in particular, that the F curves, and therefore the radial distributions, of Thomas' model[63] are in approximate accord with those actually observed. If it is true that the scattering of coherent radiation is to be calculated in all cases by the Schrödinger density distribution, we should test our model against this distribution.

An interesting attempt along these lines has been recently made by Pauling [33]. He has used certain simplifying assumptions to obtain an approximate Schrödinger density-distribution for many-electron atoms. I have shown in Fig. 11 four sets of curves. The radial electron distributions deduced by Havighurst and by Compton are shown as one curve since they are very similar. The figure shows also our first analysis of electron distribution. Matched against these are plotted the generalised distribution of the Thomas model, and the Schrödinger density distribution calculated by Pauling.

We have obviously not yet reached a point when we can be satisfied with the agreement between theory and experiment, yet the success attained so far is a distinct encouragement to further investigation.

8. — The refraction of X-rays

At Professor Lorentz's[64] suggestion I have added a very brief note on the refraction of X-rays, since the phenomenon is so intimately connected with the question of intensity of reflection and scattering, and is another example of the successful application of classical laws. The diffraction phenomenon dealt with above (intensity of reflection) arises from the scattering of coherent radiation in all directions by the atoms of a crystal. The refractive index may be considered as being due to the scattering in the forward direction of coherent radiation, which interferes with the primary beam. The arrangement of the scattering matter plays no part, so that the body may be crystalline or amorphous. The measurement of the refractive index is thus a direct measure of the amount of coherent radiation scattered in the forward direction of the incident beam.

1. Darwin [2] appears to have been first in pointing out that theory assigns a refractive index for X-rays differing from unity by about one part in a million. He predicted that a very slight departure from the law of reflection

$$n\lambda = 2d\sin\theta_0$$

would be found, the actual angle θ being given by Darwin's formula

$$\theta - \theta_0 = \frac{1-\mu}{\sin\theta\cos\theta}\ .$$

Ewald's [34] independent treatment of X-ray reflection leads to an equivalent result, though the problem is approached along quite different lines.

As is well known, the first experimental evidence of an index of refraction was found in a departure from the reflection laws. Stenström [35] observed differences in the apparent wavelength of soft X-rays (3 Å) as measured in the different orders, which were explained by Ewald's laws of X-ray reflection. The increased accuracy of X-ray spectroscopy has shown that similar deviations from the simple law of reflection exist for harder rays, though the deviations are much smaller than in the ordinary X-ray region.[65] Thus the deviations have been detected for hard rays by Duane and Patterson [36] and by Siegbahn and Hjalmar [37]. It is difficult to measure the refractive index by means of these deviations in the ordinary way, since they are so small, but Davis [38, 39] developed a very ingenious way of greatly increasing the effect. A crystal is ground so that the rays reflected by the atomic planes enter or leave a face at a very fine glancing angle, and thus suffer a comparatively great deflection.

Compton [40] discovered the total reflection of X-rays, and measured the index of refraction in this way. The refractive index is slightly less than unity, hence X-rays falling at a very fine glancing angle on a plane surface of a body are totally reflected, none of the radiation passing into the body. Compton showed that, although the refractive index is so nearly unity, yet the critical glancing angle is quite appreciable.

Finally, the direct effect of refraction by a prism has been observed by Larsson, Siegbahn and Waller [41]. X-rays entered one face of a glass prism at a very fine glancing angle, and suffered a measurable deflection. They obtained in this way a dispersion spectrum of X-rays.

2. In all cases where the frequency of the X-radiation is great compared with any frequency characteristic of the atom, the refractive index measured by any of these methods is in close accord[66] with the formula

$$1-\mu=\frac{ne^2}{2\pi m\nu^2}\ ,$$

where n is the number of electrons per unit volume in the body, e and m are the electronic constants, and ν the frequency of the incident radiation. The formula follows directly from the classical Drude-Lorentz theory of dispersion, in the limiting case where the frequency of the radiation is large compared with the 'free periods' of the electrons in the atom. It can be put in the form [42][67]

$$1-\mu=2.71\times 10^{-6}\,\frac{\rho Z}{A}\lambda^2\ ,$$

where λ is the wavelength in Ångström units of the incident radiation, ρ the density of the substance, Z and A the average atomic number and atomic weight of its constituents (for all light atoms Z/A is very nearly 0.5).[68] Expressed in this form, the order of magnitude of $1-\mu$ is easily grasped. The critical glancing angle θ for total reflection is given by

$$\cos\theta=\mu\ ,$$

whence

$$\theta=\sqrt{\frac{ne^2}{\pi m\nu^2}}\ .$$

Expressing θ in minutes of arc, and λ in Ångström units as before,

$$\theta=8.0\,\lambda\sqrt{\frac{\rho Z}{A}}\ .$$

Measurements of refractive index have been made by Compton and by Doan using the method of total reflection, by Davis, Hatley and Nardroff using reflection in a crystal, and by Larsson, Siegbahn and Waller with a prism. A variety of substances has been examined, and wavelengths between 0.5 and 2 Å have been used. The accuracy of the experimental determination of $1-\mu$ is of the order of one to five per cent. As long as the critical frequencies of the atom have not been approached, the results have agreed with the above formula within experimental error. Just as in the measurements of intensity of reflection the F curves approach a limit at small angles equal to the number of electrons in the atom, so these measurements of refractive index when interpreted by classical theory lead to a very accurate numbering of the electrons in the scattering units.

3. A highly interesting field is opened up by the measurements of refractive index for wavelengths in the neighbourhood of a critical frequency of the atom. It is a striking fact that the simple dispersion formula

$$\mu - 1 = \frac{e^2}{2\pi m} \sum_1^n \frac{n_s}{\nu_s^2 - \nu^2}$$

still gives values for the refractive index agreeing with experiment in this region, except when the critical frequency is very closely approached indeed. Davis and von Nardroff reflected $\mathrm{Cu}_{\mathrm{K}\alpha}$ and $\mathrm{Cu}_{\mathrm{K}\beta}$ X-rays[69] from iron pyrites, and found that the refractive indices could be reproduced by substituting constants in the formulae corresponding to two K electrons in iron with the frequency of the K absorption edge.[70] R. L. Doan [44] has recently made a series of measurements by the total reflection method. His accurate data support the conclusion that the Drude-Lorentz theory of dispersion represents the facts, 'not only in regions remote from the absorption edge,[71] but also in some instances in which the radiation approaches the natural frequencies of certain groups of electrons'. The existence of two K electrons[72] is very definitely indicated. Kallmann and Mark [43] have gone more deeply into the form of the dispersion curve in the neighbourhood of the critical frequencies. The change in scattering power of an atom as the frequency of the scattered radiation passes through a critical value is of course another aspect of this anomalous dispersion; the experiment of Mark and Szilard which showed this effect has been described above. There is ample evidence that measurements of refractive index will in future prove to be a most fruitful means of investigating the response of the atom to incident radiation of frequency very near each of its own characteristic frequencies.

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[43] H. Kallmann and H. Mark, *Ann. d. Physik*, **82** (1927), 585.
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A very complete account of work on intensity of reflection is given by Compton in *X-rays and Electrons* and by Ewald in volume 24 of the *Handbuch der Physik* by H. Geiger and K. Scheel,[76] *Aufbau der festen Materie und seine Erforschung durch Röntgenstrahlen*, section 18.

Discussion of Mr Bragg's report

Mr Debye. — To what extent can you conclude that there exists an energy at absolute zero?

Mr Bragg. — Waller and James have recently submitted a paper to the Royal Society in which they discuss the relation between the influence of temperature on the intensity of reflection (Debye effect) and the elastic constants of a crystal. Using the experimentally determined value of the Debye coefficient, they deduce the scattering by an atom at rest from the scattering by the atom at the temperature of liquid air (86° abs.). The curve deduced for the scattering by a perfectly motionless atom can of course take two forms, according to whether or not, in interpreting the results of the experiment, one assumes the existence of an energy at absolute zero.

If one assumes the existence of such an energy, the curve deduced from the experimental results agrees with that calculated by Hartree by applying Schrödinger's mechanics. The agreement is really very good for sodium as well as for chlorine. On the other hand, the curve that one obtains if one does not assume any energy at absolute zero deviates considerably from the calculated curve by an amount that exceeds the possible experimental error.

If these experimental results[a] are confirmed by new experiments, they provide a direct and convincing proof of the existence of an energy at absolute zero.

Mr Debye. — Would the effect not be larger if one did the experiments with diamond?

Mr Bragg. — In the case of diamond, it is difficult to interpret the results obtained using a single crystal, because the structure is very perfect and the 'extinction' is strong. One would have to work with diamond powder. But I cannot say if it would be easy to find that there exists an energy at absolute zero in diamond; I should consider it further.

Mr Fowler. — Here is how Hartree calculates the atomic fields. Starting from Thomas' atomic field, taken as a first approximation, he

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[a] *Note added 5 April 1928.* The results to which allusion is made here have just been published in detail by Messrs James, Waller and Hartree in a paper entitled: 'An investigation into the existence of zero-point energy in the rock-salt lattice by an X-ray diffraction method' (*Proc. Roy. Soc. A*, **118** (1928), 334).

calculates the Schrödinger functions for an electron placed in this field, then the density of charge in the atom corresponding to the Schrödinger functions, and then the corresponding atomic field, which will differ from that of Thomas. By successive approximations one modifies the field until the calculations yield the field which served as a starting point. This method gives very good values for the levels corresponding to X-rays and to visible light, and leads to the atom that Mr Bragg considered for comparison with experiments.

MR HEISENBERG. — How can you say that Hartree's method gives exact results, if it has not given any for the hydrogen atom? In the case of hydrogen the Schrödinger functions must be calculated with the aid of his differential equation, in which one introduces only the electric potential due to the nucleus. One would not obtain correct results if one added to this potential the one coming from a charge distribution by which one had replaced the electron. One may then obtain exact results only by taking the charge density of all the electrons, except the one whose motion one wishes to calculate. Hartree's method is certainly very useful and I have no objection to it, but it is essentially an approximation.

MR FOWLER. — I may add to what I have just said that Hartree is always careful to leave out the field of the electron itself in each state, so that, when he considers an L electron, for example, the central part of the field of the whole atom is diminished by the field of an L electron, as far as this may be considered as central. Hartree's method would then be entirely exact for hydrogen and in fact he has shown that it is extremely close to being exact for helium. (One finds a recent theoretical discussion of Hartree's method, by Gaunt, in *Proc. Cambr. Phil. Soc.*, **24** (1928), 328.)

MR PAULI. — In my opinion one must not perform the calculations, as in wave mechanics, by considering a density $|\psi(x,y,z)|^2$ in three-dimensional space,[77] but must consider a density in several dimensions

$$|\psi(x_1,\, y_1,\, z_1,\, ...,\, x_N,\, y_N,\, z_N)|^2 \ ,$$

which depends on the N particles in the atom. For sufficiently short waves the intensity of coherent scattered radiation is then proportional to[78]

$$\int \dots \int \sum_{1}^{N} e^{\frac{2\pi i}{\lambda}(\overrightarrow{n_d}-\overrightarrow{n_u},\,\overrightarrow{r_k})}\, |\psi(x_1,\ \dots,\ z_N)|^2\, dx_1 \dots dz_N\ ,$$

where λ is the wavelength of the incident radiation, $\overrightarrow{n_u}$ a unit vector in the direction of propagation, and $\overrightarrow{n_d}$ the corresponding unit vector for the scattered radiation; the sum must be taken over all the particles. The result that one obtains by assuming a three-dimensional density cannot be rigorously exact; it can only be so to a certain degree of approximation.

Mr Lorentz. — How have you calculated the scattering of radiation by a charge distributed over a region comparable to the volume occupied by the atom?

Mr Bragg. — To interpret the results of observation as produced by an average distribution of the scattering material, we applied J. J. Thomson's classical formula for the amplitude of the wave scattered by a single electron.

Mr Compton. — If we assume that there is always a constant ratio between the charge and mass of the electron, the result of the classical calculation of reflection by a crystal is exactly the same, whether the charge and mass are assumed concentrated in particles (electrons) or distributed irregularly in the atom. The intensity of reflection is determined by the average density of the electric charge in different parts of the atom. That may be represented either by the probability that a point charge occupies this region or by the volume density of an electric charge distributed in a continuous manner through this region.

Mr Kramers. — The use that one may make of the simple Thomas model of the atom in the search for the laws of reflection is extremely interesting. It would perhaps not be superfluous to investigate what result would be obtained for the electron distribution if, instead of restricting oneself to considering a single centre of attraction, one applied Thomas' differential equation to an infinity of centres distributed as in a crystal grating. Has anyone already tried to solve the problem of which Mr Bragg has just spoken, of the calculation of the general distribution of the electronic density around the nucleus of a heavy atom, in the case where there are many nuclei, as in a crystal?

Mr Bragg. — No, no one has yet attacked this problem, which I only mentioned because it is interesting.

Mr Dirac. — Do the scattering curves depend on the phase relations between the oscillations of different atoms?

Mr Bragg. — No, because the results of our experiments give only the average scattering produced in each direction by a very large number of atoms.

Mr Dirac. — What would happen if you had two simple oscillators performing harmonic vibrations? Would they produce a different scattering when in phase than when out of phase?

Mr Born. — The correct answer to the question of scattering by an atom is contained in the remark by Mr Pauli. Strictly speaking there is no three-dimensional charge distribution that may describe exactly how an atom behaves; one always has to consider the total configuration of all the electrons in the space of $3n$ dimensions. A model in three dimensions only ever gives a more or less crude approximation.

Mr Kramers asks a question concerning the influence of the Compton effect on the scattering.

Mr Bragg. — I have already said something on that subject in my report.[a] Assuming a model of the atom of the old type, Jauncey and Williams have used the criterion that the wavelength is modified when the recoil of the scattering electron is sufficient to take it entirely outside the atom. Williams was the first to apply this criterion to scattering curves obtained with crystals. He pointed out that while the speed of the recoil electron depends on both the scattering angle and the wavelength, any criterion one uses is a function of $\frac{\sin\theta}{\lambda}$, just as the interference effects depend on $\frac{\sin\theta}{\lambda}$. This implies that the existence of the Compton effect modifies the scattering curve such that we can always assign the same scattering curve to no matter what type of atom, whatever the

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[a] Cf. Bragg's report, section 6 (*eds*).

wavelength may be.

Mr Fowler. — If I have understood properly, Mr Bragg uses theoretical calculations by Waller that have not yet been published. When light is scattered by an atom in accordance with the interpretation given by Mr Waller by means of the new mechanics, the *total* amount of scattered light is given exactly by J. J. Thomson's classical formula (except for very hard γ-rays). This light is composed of the coherent scattered radiation and of the modified light (Compton scattering). In the theorem of the reflection of X-rays only the coherent scattered light must be used, and indeed it is; and this light is given exactly by the F curves like those proposed by Hartree. These F curves for atomic scattering are obviously given simply by the classical scattering for each electron, diminished by interference.

Mr Bragg. — I should like to develop Mr Fowler's remark by recalling Waller and Wentzel's conclusions briefly sketched in my report. The scattering by one of the electrons in an atom partly remains the same and partly is modified. Within certain limits the total amount of scattered radiation is given by J. J. Thomson's formula. A fraction f^2 of this amount is not modified, f being a coefficient smaller than 1, depending on the interference of the spatial distribution of the charge according to Schrödinger and calculated according to the classical laws of optics. The remaining fraction $1 - f^2$ is modified.[79]

Mr Lorentz. — It is, without doubt, extremely noteworthy that the total scattering, composed of two parts of quite different origin, agrees with Thomson's formula.

Mr Kramers makes two remarks:

1. As Mr Bragg has pointed out the importance of there being interest in having more experimental data concerning the refrangibility of X-rays in the neighbourhood of the absorption limit, I should like to draw attention to experiments performed recently by Mr Prins in the laboratory of Professor Coster at Groningen. By means of his apparatus (the details of the experiments and the results obtained are described in a paper published recently in *Zeitschrift für Physik*, **47** (1928), [479]), Mr Prins finds in a single test the angle of total reflection corresponding to an extended region of frequencies. In the region of the absorption limit of the metal,

he finds an abnormal effect, which consists mainly of a strong decrease in the angle of total reflection on the side of the absorption limit located towards the short wavelengths. This effect is easily explained taking into account the influence of absorption on the total reflection, without it being necessary to enter into the question of the change in refrangibility of the X-rays. In fact, the absorption may be described by considering the refractive index n as a complex number, whose imaginary part is related in a simple manner to the absorption coefficient. Introducing this complex value for n in the well-known formulas of Fresnel for the intensity of reflected rays, one finds that the sharp limit of total reflection disappears, and that the manner in which the intensity of reflected rays depends on the angle of incidence is such that the experiment must give an 'effective angle of total reflection' that is smaller than in the case where there is no absorption and that decreases as the absorption increases.

According to the atomic theory one would also expect to find, in the region of the absorption limit, anomalies in the real part of the refractive index, producing a similar though less noticeable decrease of the effective angle of total reflection on the side of the absorption edge directed towards the large wavelengths. Mr Prins has not yet succeeded in showing that the experiments really demonstrate this effect.[a]

The theory of these anomalies in the real part of the refractive index constitutes the subject of my second remark.

2. Let us consider plane and polarised electromagnetic waves, in which the electric force can be represented by the real part of $Ee^{2\pi i\nu t}$, striking an atom which for further simplicity we shall assume to be isotropic. The waves make the atom behave like an oscillating dipole, giving, by expansion in a Fourier series, a term with frequency ν. Let us represent this term by the real part of $Pe^{2\pi i\nu t}$, where P is a complex vector having the same direction as the vector E to which it is, moreover, proportional. If we set

$$\frac{P}{E} = f + ig \ , \tag{1}$$

where f and g are real functions of ν, the real and imaginary parts of the refractive index of a sample of matter are related in a simple way to the functions f and g of the atoms contained in the sample.

Extending the domain of values that ν may take into the negative

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[a] Continuing his research Mr Prins has established (February 1928) the existence of this effect, in agreement with the theory.

region and defining f as an even function of ν, g as an odd function, one easily verifies that the dispersion formulas of Lorentz's classical theory and also those of modern quantum mechanics are equivalent to the formula

$$f(\nu) = \frac{1}{\pi} ⨍_{-\infty}^{+\infty} \frac{g(\nu')}{\nu - \nu'} d\nu' \ , \tag{2}$$

where the sign $⨍$ indicates the 'principal' value of the integral.

This formula can easily be applied to atoms showing continuous absorption regions and is equivalent to the formulas proposed for these cases by R. de Laer Kronig and by Mark and Kallmann. There is hardly any doubt that this general formula may be derived from quantum mechanics, if one duly takes into account the absorption of radiation, basing oneself on Dirac's theory, for example.

From a mathematical point of view, formula (2) gives us the means to construct an analytic function of a complex variable ν that is holomorphic below the real axis and whose real part takes the values $g(\nu')$ on this axis. If one considers ν as a real variable, the integral equation (2) has the solution

$$g(\nu) = -\frac{1}{\pi} ⨍_{-\infty}^{+\infty} \frac{f(\nu')}{\nu - \nu'} d\nu' \ , \tag{3}$$

which shows that the imaginary part of the refractive index depends on the real part in nearly the same way as the real part depends on the imaginary part. The fact that the analytic function f of the complex variable ν, defined by (2) for the lower half of the complex plane, has no singularity in this half-plane, means that dispersion phenomena, when one studies them by means of waves whose amplitude grows in an exponential manner (ν complex), can never give rise to singular behaviour for the atoms.

MR COMPTON. — The measurements of refractive indices of X-rays made by Doan agree better with the Drude-Lorentz formula than with the expression derived by Kronig based on the quantum theory of dispersion.

MR DE BROGLIE. — I should like to draw attention to recent experiments carried out by Messrs J. Thibaud and A. Soltan,[a] which touch on the questions raised by Mr Bragg. In these experiments Messrs

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[a] *C. R. Acad. Sc.*, **185** (1927), 642.

Thibaud and Soltan measured, by the tangent grating method, the wavelength of a certain number of X-rays in the domain 20 to 70 Å. Some of these wavelengths had already been determined by Mr Dauvillier using diffraction by fatty-acid gratings. Now, comparing the results of Dauvillier with those of Thibaud and Soltan, one notices that there is a systematic discrepancy between them that increases with wavelength. Thus for the K_α line of boron, Thibaud and Soltan find 68 Å, while Dauvillier had found 73.5 Å, that is, a difference of 5.5 Å. This systematic discrepancy appears to be due to the increase of the refractive index with wavelength. The index does not actually play a role in the tangent grating method, while it distorts in a systematic way the results obtained by crystalline diffraction when one uses the Bragg formula. Starting from the difference between their results and those of Mr Dauvillier, Messrs Thibaud and Soltan have calculated the value of the refractive index of fatty acids around 70 Å and found

$$\delta = 1 - \mu = 10^{-2}$$

thereabouts. This agrees well with a law of the form $\delta = K\lambda^2$; since in the ordinary X-ray domain the wavelengths are about 100 times smaller, δ is of order 10^{-6}. One could object that, according to the Drude-Lorentz law, the presence of K discontinuities of oxygen, nitrogen and carbon between 30 and 45 Å should perturb the law in λ^2. But in the X-ray domain the validity of Drude's law is doubtful, and if one uses in its place the formula proposed by Kallmann and Mark[a] the agreement with the experimental results is very good. Let us note finally that the existence of an index appreciably different from 1 can contribute to explaining why large-wavelength lines, obtained with a fatty-acid grating, are broad and spread out.

Mr Lorentz makes a remark concerning the refractive index of a crystal for Röntgen rays and the deviations from the Bragg law. It is clear that, according to the classical theory, the index must be less than unity, because the electrons contained in the atoms have eigenfrequencies smaller than the frequency of the rays, which gives rise to a speed of propagation greater than c. But in order to speak of this speed, one must adopt the macroscopic point of view, abstracting away the molecular discontinuity. Now, if one wishes to explain Laue's phenomenon in all its details, one must consider, for example, the action of the vibrations

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[a] *Ann. d. Phys.*, **82** (1927), 585.

excited in the particles of a crystallographic layer on a particle of a neighbouring layer. This gives rise to series that one cannot replace by integrals. It is for this reason that I found some difficulty in the explanation of deviations from the Bragg law.[80]

Mr Debye. — Ewald has tried to do similar calculations.

Mr Lorentz. — It is very interesting to note that with Röntgen rays one finds, in the vicinity of an absorption edge, phenomena similar to those that in classical optics are produced close to an absorption band. There is, however, a profound difference between the two cases, the absorption edge not corresponding to a frequency that really exists in the particles.

Notes to the translation

1 Here and in a few other places, the French adds (or omits) inverted commas.
2 [réflexion des radiations des raies]
3 [ne fait prévoir aucun écart]
4 The French edition adds 'en couches' [in layers].
5 Typescript: 'have often been .... it has led'; French version: 'a souvent été .... elles ont conduit'.
6 [ont eu confiance dans les méthodes quantitatives]
7 [portions]
8 [sous une forme plus primitive]
9 [il soit possible de les trouver]
10 [logiquement]
11 [thermodynamique]
12 Word omitted in the French version.
13 The French edition adds 'Sir'.
14 [les données obtenues par réflexion]
15 Here following the French edition; the typescript reads 'total radiation'.
16 Emphasis omitted in the French edition.
17 [servirent à comparer]
18 The French omits 'of reflection'.
19 [déjà observée]
20 [il proposa d'employer, pour l'interprétation des mesures, la méthode de Fourier]
21 [rapport]
22 [ou]
23 [Mc Innes]
24 [potassium]
25 The typescript has a spurious comma after 'Bergen'.
26 [à l'aide de cristaux]
27 [les deux constantes électroniques]
28 Here and in some other instances, the French renders 'single' as 'simple'.
29 ['facteur atomique']
30 Not printed as a quotation in the French edition.
31 choisies simplement
32 The French adds 'de ses expériences'.
33 The French adds 'dans ces conditions'.
34 This is indeed a quotation from p. 272 from the lecture by W. H. Bragg. The typescript has a comma instead of the closing quotation mark, while the French edition omits the opening quotation mark. The typescript has a spurious denominator 'a' instead of 'd' in the second and third terms (but tacitly corrects another typo in the original).
35 [dans son livre sur 'les rayons X et les électrons']
36 The French adds 'on peut procéder de la façon inverse, c'est-à dire'.
37 'Beryl.' omitted in French edition.
38 The French edition uses superscripts throughout.
39 The French edition omits the overbar in the caption.
40 [Wyckhoff]
41 Word omitted in French edition.
42 Again, the French edition omits the overbar in the caption.

43 [frappante]
44 [son livre sur les rayons X et les électrons]
45 [bien que les figures (*sic*) sur lesquelles la nouvelle courbe se base soient plus exactes]
46 Here and in several other places, the French adds ‘Mlle’.
47 The French reads ‘une énergie au zéro absolu (énergie de structure)’.
48 Word missing in the French edition.
49 Bracket printed as a footnote in the French edition.
50 [tout près de ce point]
51 [irrégularités]
52 [pour]
53 [brillante]
54 [λ $\mathrm{Sr}_{\mathrm{K}\alpha} = 0.871\text{Å}$]
55 [faites d’après 1921 figures]
56 The French omits ‘B. J. B. figures for NaCl’.
57 [figures]
58 Again, in the French, the false friend ‘figures’.
59 [déduits]
60 [CaFl, LiFl, NaFl]
61 [Une fois que nous disposerons de données précises]
62 The French edition omits ‘& Compton’ and has ‘Modèle de Pauling et Schrödinger’.
63 The French translates as if the comma were after ‘of Thomas’ model’ rather than before.
64 [M. Lorentz]
65 The typescript reads ‘much smaller in the ordinary X-ray region’, but given the context the text should be amended as shown (as also done in the French version).
66 [parfaitement d’accord]
67 Reference omitted in the French edition.
68 [$\frac{Z}{A}$ la valeur moyenne du rapport du nombre atomique au poids atomique pour ses divers constituants (pour tous les atomes légers ce rapport est à peu près égal à 0.5]
69 [rayons]
70 Typescript: ‘of the K adsorption edge’; French version: ‘de la discontinuité K’.
71 Typescript: ‘adsorption edge’; French: ‘bord d’absorption’.
72 The French adds ‘dans la pyrite’.
73 French edition: ‘Mac Innes’.
74 Typescript and French edition both have ‘532’.
75 Both typescript and French edition give this reference as ‘B. Davis and C. C. Hatley’. The typescript has ‘291’.
76 Authors added in the French edition.
77 Here and in the following displayed formula, the published version has square brackets instead of absolute bars.
78 Arrow missing on $\overrightarrow{r_k}$ in the published volume.
79 The original text mistakenly states that both fractions are ‘not modified’.
80 The mixing of first and third person, here and in a few similar instances throughout the discussions, is as in the published text.

Disagreements between experiment and the electromagnetic theory of radiation[a]

By Mr Arthur H. COMPTON

Introduction

Professor W. L. Bragg has just discussed a whole series of radiation phenomena in which the electromagnetic theory is confirmed. He has even dwelt on some of the limiting cases, such as the reflection of X-rays by crystals, in which the electromagnetic theory of radiation gives us, at least approximately, a correct interpretation of the facts, although there are reasons to doubt that its predictions are truly exact. I have been left the task of pleading the opposing cause to that of the electromagnetic theory of radiation, seen from the experimental viewpoint.

I have to declare from the outset that in playing this role of the accuser I have no intention of diminishing the importance of the electromagnetic theory as applied to a great variety of problems.[b] It is, however, only by

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[a] An English version of this report (Compton 1928) was published in the *Journal of the Franklin Institute*. The French version appears to be essentially a translation of the English paper with some additions. Whenever there are no discrepancies, we reproduce Compton's own English (we have corrected some obvious typos and harmonised some of the spelling). Interesting variants are footnoted. Other discrepancies between the two versions are reported in the endnotes (*eds.*).

[b] The opening has been translated from the French edition. The English version has the following different opening (*eds.*):

During the last few years it has become increasingly evident that the classical electromagnetic theory of radiation is incapable of accounting for certain large classes of phenomena, especially those concerned with the interaction between radiation and matter. It is not that we question the wave character of light — the striking successes of this conception in explaining polarisation and interference of light can leave no doubt that radiation has the characteristics acquainting ourselves with the real or apparent[1] failures of this powerful theory that we can hope to develop a more complete theory of radiation which will describe the facts as we know them.

The more serious difficulties which present themselves in connection with the theory that radiation consists of electromagnetic waves, propagated through space in accord with the demands of Maxwell's equations, may be classified conveniently under five heads:[2]

(1) Is there an ether? If there are oscillations, there must be a medium in which these oscillations are produced. Assuming the existence of such a medium, however, one encounters great difficulties.

(2) How are the waves produced? The classical electrodynamics requires as a source of an electromagnetic wave an oscillator of the same frequency as that of the waves it radiates. Our studies of spectra,[3] however, make it appear impossible that an atom should contain oscillators of the same frequencies as the emitted rays.

(3) The photoelectric effect. This phenomenon is wholly anomalous when viewed from the standpoint of waves.

(4) The scattering of X-rays, and the recoil electrons, phenomena in which we find gradually increasing departures from the predictions of the classical wave theory as the frequency increases.

(5) Experiments on individual interactions between quanta of radiation and electrons. If the results of the experiments of this type are reliable, they seem to show definitely that individual quanta of radiation, of energy $h\nu$, proceed in definite directions.

The photon hypothesis.[4] — In order to exhibit more clearly the difficulties with the classical theory of radiation, it will be helpful to keep in mind the suggestion that[5] light consists of corpuscles. We need not think of these two views as necessarily alternative. It may well be that the two conceptions are complementary. Perhaps the corpuscle is related

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of waves; but it is equally true that certain other properties of radiation are not easily interpreted in terms of waves. The power of the electromagnetic theory as applied to a great variety of problems of radiation is too well known to require emphasis.

to the wave in somewhat the same manner that the molecule is related to matter in bulk; or there may be a guiding wave which directs the corpuscles which carry the energy. In any case, the phenomena which we have just mentioned suggest the hypothesis that radiation is divisible into units possessing energy $h\nu$, and which proceed in definite directions with momentum $h\nu/c$. This is obviously similar to Newton's old conception of light corpuscles. It was revived in its present form by Professor Einstein,[6] it was defended under the name of the 'Neutron Theory' by Sir William [H.] Bragg, and has been given new life by the recent discoveries associated with the scattering of X-rays.

In referring to this unit of radiation I shall use the name 'photon', suggested recently by G. N. Lewis.[a] This word avoids any implication regarding the nature of the unit, as contained for example in the name 'needle ray'. As compared with the terms 'radiation quantum' and 'light quant',[7] this name has the advantages of brevity and of avoiding any implied dependence upon the much more general quantum mechanics or quantum theory of atomic structure.

Virtual radiation. — Another conception of the nature of radiation which it will be desirable to compare with the experiments is Bohr, Kramers and Slater's important theory of virtual radiation.[b] According to this theory, an atom in an excited state is continually emitting virtual radiation, to which no energy characteristics are to be ascribed. The normal atoms have associated with them virtual oscillators, of the frequencies corresponding to jumps of the atom to all of the stationary states of higher energy. The virtual radiation may be thought of as being absorbed by these virtual oscillators, and any atom which has a virtual oscillator absorbing this virtual radiation has a certain probability of jumping suddenly to the higher state of energy corresponding to the frequency of the particular virtual oscillator. On the average, if the radiation is completely absorbed, the number of such jumps to levels of higher energy is equal to the number of emitting atoms which pass from higher to lower states. But there is no direct connection between the falling of one atom from a higher to a lower state and a corresponding rise of a second atom from a lower to a higher state. Thus on this view the energy of the emitting atoms and of the absorbing atoms is only statistically conserved.

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[a] G. N. Lewis, *Nature*, [**118**], [874] (Dec. 18, 1926).

[b] N. Bohr, H. A. Kramers and J. C. Slater, *Phil. Mag.*, **47** (1924), 785; *Zeits. f. Phys.*, **24** (1924), 69.

THE PROBLEM OF THE ETHER[8]

The constancy of the speed of radiation of different wavelengths has long been considered as one of the most powerful arguments in favour of the wave theory of light. This constancy suggests that a perturbation is travelling through a fixed medium in space, the ether.

If experiments like those by Michelson and Morley's were to show the existence of a relative motion with respect to such a medium, this argument would be considerably strengthened. For then we could imagine light as having a speed determined with reference to a fixed axis in space. But, except for the recent and quite doubtful experiments by Miller,[a] no-one has ever detected such a relative motion. We thus find ourselves in the difficult position of having to imagine a medium in which perturbations travel with a definite speed, not with reference to a fixed system of axes, but with reference to each individual observer, whatever his motion. If we think of the complex properties a medium must have in order to transmit a perturbation in this way, we find that the medium differs so considerably from the simple ether from which we started that the analogy between a wave in such a medium and a pertubation travelling in an elastic medium is very distant. It is true that doubts have often been expressed as to the usefulness of retaining the notion of the ether. Nevertheless, if light is truly a wave motion, in the sense of Maxwell, there must be a medium in order to transmit this motion, without which the notion of wave would have no meaning. This means that, instead of being a support for the wave theory, the concept of the ether has become an uncomfortable burden of which the wave theory has been unable to rid itself.

If, on the other hand, we accept the view suggested by the theory of relativity, in which for the motion of matter or energy there is a limiting speed relative to the observer, it is not surprising to find a form of energy that moves at this limiting speed. If we abandon the idea of an ether, it is simpler to suppose that this energy moves in the form of corpuscles rather than waves.

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[a] D. C. Miller, *Nat. Acad. Sci. Proc.*, **11** (1925), 306.

THE EMISSION OF RADIATION

When we trace a sound to its origin, we find it coming from an oscillator vibrating with the frequency of the sound itself. The same is true of electric waves, such as radio waves, where the source of the radiation is a stream of electrons oscillating back and forth in a wire. But when we trace a light ray or an X-ray back to its origin, we fail to find any oscillator which has the same frequency as the ray itself. The more complete our knowledge becomes of the origin of spectrum lines, the more clearly we see that if we are to assign any frequencies to the electrons within the atoms, these frequencies are not the frequencies of the emitted rays, but are the frequencies associated with the stationary states of the atom. This result cannot be reconciled with the electromagnetic theory of radiation, nor has any mechanism been suggested whereby radiation of one frequency can be excited by an oscillator of another frequency. The wave theory of radiation is thus powerless to suggest how the waves originate.

The origin of the radiation is considerably simpler when we consider it from the photon viewpoint. We find that an atom changes from a stationary state of one energy to a state of less energy, and associated with this change radiation is emitted. What is simpler than to suppose that the energy lost by the atom is radiated away as a single photon? It is on this view unnecessary to say anything regarding the frequency of the radiation. We are concerned only with the energy of the photon, its direction of emission, and its state of polarisation.

The problem of the emission of radiation takes an especially interesting form when we consider the production of the continuous X-ray spectrum.[a] Experiment shows that both the intensity and the average frequency of the X-rays emitted at angles less than 90 degrees with the cathode-ray stream are greater than at angles greater than 90 degrees. This is just what we should expect due to the Doppler effect if the X-rays are emitted by a radiator moving in the direction of the cathode rays. In order to account for the observed dissymmetry between the rays in the forward and backward directions, the particles emitting the radiation must be moving with a speed of the order of 25 per cent that of light. This means that the emitting particles must be free electrons, since it would require an impossibly large energy to set an atom into motion with such a speed.

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[a] The difficulty here discussed was first emphasised by D. L. Webster, *Phys. Rev.*, **13** (1919), 303.

But it will be recalled that the continuous X-ray spectrum has a sharp upper limit. Such a sharp limit is, however, possible on the wave theory only in case the rays come in trains of waves of considerable length, so that the interference between the waves in different parts of the train can be complete at small glancing angles of reflection from the crystal. This implies that the oscillator which emits the rays must vibrate back and forth with constant frequency a large number of times while the ray is being emitted. Such an oscillation might be imagined for an electron within an atom; but it is impossible for an electron moving through an irregular assemblage of atoms with a speed comparable with that of light.

Thus the Doppler effect in the primary X-rays demands that the rays shall be emitted by rapidly moving electrons, while the sharp limit to the continuous spectrum requires that the rays be emitted by an electron bound within an atom.

The only possible escape from this dilemma on the wave theory is to suppose that the electron is itself capable of internal oscillation of such a character as to emit radiation. This would, however, introduce an undesirable complexity into our conception of the electron, and would ascribe the continuous X-rays to an origin entirely different from that of other known sources of radiation.

Here again the photon theory affords a simple solution. It is a consequence of Ehrenfest's adiabatic principle[a] that photons emitted by a moving radiator will show the same Doppler effect, with regard to both frequency and intensity, as does a beam of waves.[b] But if we suppose that photons are radiated by the moving cathode electrons, the energy of each photon will be the energy lost by the electron, and the limit of the X-ray spectrum is necessarily reached when the energy of the photon is equal to the initial energy of the electron, i.e., $h\nu = eV$. In this case, if we consider the initial state as an electron approaching an atom with large kinetic energy and the final state as the electron leaving the atom with a smaller kinetic energy, we see that the emission of the continuous X-ray spectrum is the same kind of event as the emission of any other type of radiation.

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[a] The adiabatic principle consists in the following. Since for a quantised quantity there should be no quantum jumps induced by an infinitely slowly varying external force (in this case, one that gently accelerates a radiator), there is an analogy between these quantities and the classical adiabatic invariants. Ehrenfest (1917) accordingly formulated a principle identifying the classical quantities to be quantised as the adiabatic invariants of a system (*eds.*).

[b] Cf., e.g., A. H. Compton, *Phys. Rev.*, **21** (1923), 483.

Absorption of radiation. — According to the photon theory, absorption occurs when a photon meets an atom and imparts its energy to the atom. The atom is thereby raised to a stationary state of higher energy — precisely the reverse of the emission process.

On the wave theory, absorption is necessarily a continuous process, if we admit the conservation of energy, since on no part of the wave front is there enough energy available to change the atom suddenly from a state of low energy to a state of higher energy. What evidence we have is, however, strongly against the atom having for any considerable length of time an energy intermediate between two stationary states; and if such intermediate states cannot exist, the gradual absorption of radiation is not possible. Thus the absorption of energy from waves[9] is irreconcilable with the conception of stationary states.

We have seen that on the theory of virtual radiation the energy of the emitting atoms and of the absorbing atoms is only statistically conserved. There is according to this view therefore no difficulty with supposing that the absorbing atom suddenly jumps to a higher level of energy, even though it has not received from the radiation as much energy as is necessary to make the jump. It is thus possible through virtual oscillators and virtual radiation to reconcile the wave theory of radiation with the sudden absorption of energy, and hence to retain the idea of stationary states.

THE PHOTOELECTRIC EFFECT

It is well known that the photon hypothesis was introduced by Einstein to account for the photoelectric effect.[a] The assumption that light consists of discrete units which can be absorbed by atoms only as units, each giving rise to a photoelectron, accounted at once for the fact that the number of photoelectrons is proportional to the intensity of the light; and the assumption that the energy of the light unit is equal to $h\nu$, where h is Planck's constant, made it possible to predict the kinetic energy with which the photoelectrons should be ejected, as expressed by Einstein's well-known photoelectric equation,

$$mc^2\Big(\frac{1}{\sqrt{1-\beta^2}}-1\Big)=h\nu-w_p\ . \qquad (1)$$

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[a] A. Einstein, *Ann. d. Phys.*, **17** (1905), [132].[10]

Seven years elapsed before experiments by Richardson and Compton[a] and by Hughes[b] showed that the energy of the emitted electrons was indeed proportional to the frequency less a constant,[12] and that the factor of proportionality was close to the value of h calculated from Planck's radiation formula. Millikan's more recent precision photoelectric experiments with the alkali metals[c] confirmed the identity of the constant h in the photoelectric equation with that in Planck's radiation formula. De Broglie's beautiful experiments[d] with the magnetic spectrograph showed that in the region of X-ray frequencies the same equation holds, if only we interpret the work function w_p as the work required to remove the electron from the pth energy level of the atom. Thibaud has made use of this result[e] in comparing the velocities of the photoelectrons ejected by γ-rays from different elements, and has thus shown that the photoelectric equation (1) holds with precision even for β-rays of the highest speed. Thus from light of frequency so low that it is barely able to eject photoelectrons from metals to γ-rays that eject photoelectrons with a speed almost as great as that of light, the photon theory expresses accurately the speed of the photoelectrons.

The direction in which the photoelectrons are emitted is no less instructive than is the velocity. Experiments using the cloud expansion method, performed[13] by C. T. R. Wilson[a] and others,[b] have shown that the most probable direction in which the photoelectron is ejected from an atom is nearly the direction of the electric vector of the incident wave, but with an appreciable forward component to its motion. There is, however, a very considerable variation in the direction of emission. For example, if we plot the number of photoelectrons ejected at different angles with the primary beam we find, according to Auger, the distribution shown in Fig. 1.

Each of these curves, taken at a different potential, represents the distribution of about 200 photoelectron tracks. It will be seen that as the potential on the X-ray tube increases, the average forward component of the photoelectron's motion also increases.

When polarised X-rays are used, there is a strong preponderance of the

[a] O. W. Richardson and K. T. Compton, *Phil. Mag.*, **24** (1912), 575.

[b] A. L. Hughes, *Phil. Trans. A*, **212** (1912), 205.[11]

[c] R. A. Millikan, *Phys. Rev.*, **7** (1916), 355.

[d] M. de Broglie, *Jour. de Phys.*, **2** (1921), 265.

[e] J. Thibaud, *C. R.*, **179** (1924), 165, 1053 and 1322.

[a] C. T. R. Wilson, *Proc. Roy. Soc. A*, **104** (1923), 1.

[b] A. H. Compton, *Bull. Natl. Res. Coun.*, No. 20 (1922), 25; F. W. Bubb, *Phys. Rev.*, **23** (1924), 137; P. Auger, *C. R.*, **178** (1924), 1535; D. H. Loughridge, *Phys. Rev.*, **26** (1925), 697; F. Kirchner, *Zeits. f. Phys.*, **27** (1926), 385.

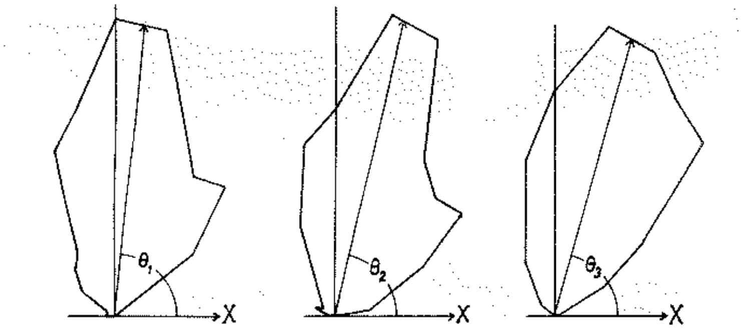

Fig. 1. Longitudinal distribution of photoelectrons for X-rays of three different effective wavelengths, according to Auger.

photoelectrons in or near the plane including the electric vector of the incident rays. Thus Fig. 2 shows the distribution found by Bubb of the direction of the photoelectrons ejected from moist air when traversed by X-rays that have been polarised by scattering at right angles from a block of paraffin. Because of multiple scattering in the paraffin, the scattered rays are not completely polarised, and this is probably sufficient to account for the fact that some photoelectrons appear to start at right angles with the electric vector. This effect with X-rays is doubtless similar in character to the selective photoelectric effect discovered many years ago by Pohl and Pringsheim, in which the number of electrons ejected by light from the liquid surface of sodium-potassium alloy is greater when the electric vector is in a plane perpendicular to the surface than when parallel to the surface.

Recent experiments have shown that the direction in which the photoelectrons are ejected by X-rays is at least very nearly independent of the material from which the electrons come.[a]

Can electromagnetic waves produce photoelectrons? — Before discussing the production of photoelectrons from the standpoint of radiation quanta, let us see what success meets the attempt[15] to account for them on the basis of electromagnetic waves. The fact that they are emitted

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[a] E. A. Owen, *Proc. Phys. Soc.*, **30** (1918), 133; Auger, Kirchner, Loughridge, *loc. cit.*[14]

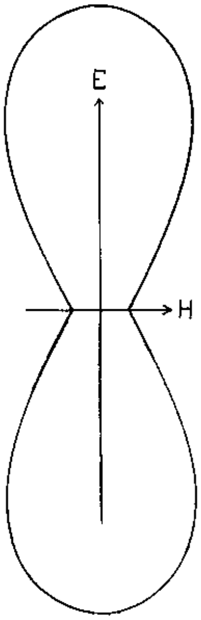

Fig. 2. Lateral distribution of photoelectrons for incompletely polarised X-rays, according to Bubb.

approximately in the direction of the electric vector would suggest that the photoelectrons are ejected by the direct action of the electric field of the incident rays. If this were the case, however, we should expect the speed of the ejected electrons to be greater for greater intensity of radiation, whereas experiment shows that for the same wavelength intense sunlight ejects an electron no faster than does the feeble light from a star. Furthermore, the energy available from the electromagnetic wave is wholly inadequate. Thus in a recent experiment performed by Joffe and Dobronrawov,[a] X-rays were produced by the impact on a target of 10^4 to 10^5 electrons per second. Since on the electromagnetic theory an X-ray pulse is of the order of 10^3 waves in length or 10^{-16} seconds

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[a] A. Joffe and N. Dobronrawov, *Zeits. f. Phys.*, **34**, 889 (1925).

in duration, the X-ray pulses must have followed each other at widely separated intervals. It was found, however, that photoelectrons were occasionally ejected from a bismuth particle which subtended a solid angle not greater than 10^{-5}. It is clearly impossible that all the energy of an X-ray pulse which has spread out in a spherical wave should spend itself on this bismuth particle. Thus on the wave theory the ejection of the photoelectron, which has almost as much energy as the original cathode electron, could not have been accomplished by a single[16] pulse. It cannot therefore be the direct action of the electric vector of the wave, taken in the usual sense,[17] which has ejected the electron.

We may assume, on the other hand, that the energy is gradually absorbed in the bismuth particle of Joffe's experiment until an amount $h\nu$ has accumulated, which is then spent in ejecting the photoelectron. We have already called attention to the fact that this gradual absorption hypothesis implies the existence of stationary states in the atom having infinitesimal gradations of energy, whereas the evidence is very strong that atoms cannot endure except in certain definitely defined stationary states. But new difficulties also arise. Why do the photoelectrons tend to start in the direction of the electric field of the incident wave? If we suppose that it is the gradual absorption of energy from a wave which liberates the electron, why does there exist a tendency for the electron to start with a large component of its motion in a forward direction?[18] The forward impulse due to the radiation pressure as[19] the energy is gradually absorbed will be transferred to the atom and not left with [the] absorbing electron. The accumulation hypothesis is thus difficult to defend.

Photons and photoelectrons. — On the photon theory it is possible to account in a simple manner for most of the properties of the photoelectrons. We have seen how Einstein was able to predict accurately the velocity of the photoelectrons, assuming only that energy is conserved when a photon acts on an electron. In order to account for the direction of emission we must ascribe to the photon some of the properties of an electromagnetic pulse. Bubb introduced the suggestion[a] that we ascribe to the photon a vector property similar to the electric vector of an electromagnetic wave, so that when the photon traverses an atom the electrons and the nucleus receive impulses in opposite directions perpendicular to the direction of propagation. Associated with this electric vector, we

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[a] F. W. Bubb, *Phys. Rev.*, **23** (1924), 137.

should also expect to find a magnetic vector. Thus if an electron is set in motion by the electric vector of the photon at right angles to the direction of propagation, the magnetic vector of the photon will act on the moving electron in the direction of propagation. This is strictly analogous to the radiation pressure exerted by an electromagnetic wave on an electron which it traverses, and means that the forward momentum of the absorbed photon is transferred to the photoelectron.

In the simplest case, where we neglect the initial momentum of the electron in its orbital motion in the atom, the angle between the direction of the incident ray and the direction of ejection is found from these assumptions to be

$$\theta = \tan^{-1} \sqrt{2/\alpha} \, , \tag{2}$$

where $\alpha = \gamma/\lambda$, and $\gamma = h/mc = 0.0242$ Å. The quantity α is small compared with unity, except for very hard X-rays and γ-rays. Thus for light, equation (2) predicts the expulsion of photoelectrons at nearly 90 degrees. This is in accord with the rather uncertain data which have been obtained with visible and ultra-violet light.[a]

The only really significant test of this result is in its application to X-ray photoelectrons. In Fig. 1 are drawn the lines θ_1, θ_2 and θ_3 for the three curves, at the angles calculated by Auger from equation (2). It will be seen that they fall very satisfactorily in the direction of maximum emission of the photoelectrons. Similar results have been obtained by other investigators.[b] This may be taken as proof that a photon imparts not only its energy, but also its momentum to the photoelectrons.[c]

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[a] Cf. A. Partsch and W. Hallwachs, *Ann. d. Phys.*, **41** (1913), 247.

[b] W. Bothe, *Zeits. f. Phys.*, **26** (1925), 59; F. Kirchner, *Zeits. f. Phys.*, **27** (1926), 385.[20]

[c] The English version includes here the following footnote. Cf. also the comments by Bragg on p. 356 and the ensuing discussion (*eds.*).

Since this was written, experiments by [D. H.] Loughridge (*Phys. Rev.*, **30** (1927), [488]) have been published which show a forward component to the photoelectron's motion which seems to be greater than that predicted by equation (2). Williams, in experiments as yet unpublished, finds that the forward component is almost twice as great as that predicted by this theory. These results indicate that the mechanism of interaction between the photon and the atom must be more complex than here postulated. The fact that the forward momentum of the photoelectron is found to be of the same order of magnitude as that of the incident photon, however, suggests that the momentum of the photon is acquired by the photoelectron, while an additional forward impulse is imparted by the atom. Thus these more recent experiments

Honesty[a] obliges me to point out a difficulty that arises in this explanation of the motion of the photoelectrons. It is the failure of the attempts made to account properly for the fact that the photoelectrons are emitted over a wide range of angles instead of in a definite direction, as would be suggested by the calculation just outlined. The most interesting of these attempts is that of Bubb,[b] who takes into account the momentum of the electron immediately before the absorption of the photon. Bubb finds a dispersion of the directions of emission of the photoelectrons of the correct order of magnitude, but which is larger when the electron issues from a heavy atom than when it issues from a light one. We have seen, however, that experiment has shown this dispersion of the directions of emission to be notably independent of the element from which the photoelectron originates.

Whatever may be the cause of the dispersion in the directions of motion of the photoelectrons,[21] it will readily be seen that if the time during which the photon exerts a force on the electron is comparable with the natural period of the electron[22] in the atom, the impulse imparted to the electron will be transferred in part to the positive nucleus about which the electron is moving. The fact that the photoelectrons are ejected with a forward component equal, within the limits of experimental error, to the momentum of the incident photon[23] means that no appreciable part of the photon's momentum is spent on the remainder of the atom. This can only be the case if the time of action of the photon on the electron is short compared with the time of revolution of the electron in its orbit.[a]

also support the view that the photoelectron acquires both the energy and the momentum of the photon.

[a] This paragraph is present only in the French edition. The corresponding one in the English edition reads:

If the angular momentum of the atomic system from which the photoelectron is ejected is to be conserved when acted upon by the radiation, the electron cannot be ejected exactly in the direction of θ, but must receive an impulse in a direction determined by the position of the electron in the atom at the instant it is traversed by the photon.* Thus we should probably consider the electric vector of the X-ray wave as defining merely the most probable direction in which the impulse should be imparted to the electron. This is doubtless the chief reason why the photoelectrons are emitted over a wide range of angles instead of in a definite direction, as would be suggested by the calculation just outlined.

With the footnote: *Cf. A. H. Compton, *Phys. Rev.*, [**31**] (1928), [59] (*eds.*).

[b] F. W. Bubb, *Phil. Mag.*, **49** (1925), 824.

[a] The English edition includes the further sentence: 'Such a short duration of

The photoelectric effect and virtual radiation. — It is to be noted that none of these properties of the photoelectron is inconsistent with the virtual radiation theory of Bohr, Kramers and Slater. The difficulties which applied to the classical wave theory do not apply here, since the energy and momentum are conserved only statistically. There is nothing in this theory, however, which would enable us to predict anything regarding the motion of the photoelectrons. The degree of success that has attended the application of the photon hypothesis to the motion of these electrons has come directly from the application of the conservation principles to the individual action of a photon on an electron. The power of these principles as applied to this case is surprising if the assumption is correct that they are only statistically valid.

PHENOMENA ASSOCIATED WITH THE SCATTERING OF X-RAYS

As is now well known, there is a group of phenomena associated with the scattering of X-rays for which the classical wave theory of radiation fails to account. These phenomena may be considered under the heads of: (1) The change of wavelength of X-rays due to scattering, (2) the intensity of scattered X-rays, and (3) the recoil electrons.

The earliest experiments on secondary X-rays and γ-rays[24] showed a difference in the penetrating power of the primary and the secondary rays. In the case of X-rays, Barkla and his collaborators[a] showed that the secondary rays from the heavy elements consisted largely of fluorescent radiations characteristic of the radiator, and that it was the presence of these softer rays which was chiefly responsible for the greater absorption of the secondary rays. When later experiments[b] showed a measurable difference in penetration even for light elements such as carbon, from which no fluorescent K or L radiation appears, it was natural to ascribe[c] this difference to a new type of fluorescent radiation, similar to the K and L types, but of shorter wavelength. Careful absorption measurements[d] failed, however, to reveal any critical absorption limit for these assumed

interaction is a natural consequence of the photon conception of radiation, but is quite contrary to the consequences of the electromagnetic theory' (*eds.*).

[a] C. [G.] Barkla and C. A. Sadler, *Phil. Mag.*, **16**, 550 (1908).[25]

[b] C. A. Sadler and P. Mesham, *Phil. Mag.*, **24** (1912), 138; J. Laub, *Ann. d. Phys.*, **46** (1915), 785.

[c] [C. G.] Barkla and [M. P.] White, *Phil. Mag.*, **34** (1917), 270; J. Laub, *Ann. d. Phys.*, **46** (1915), 785, *et al.*

[d] E.g., [F. K.] Richtmyer and [K.] Grant, *Phys. Rev.*, **15** (1920), 547.

'J' radiations similar to those corresponding to the K and L radiations. Moreover, direct spectroscopic observations[a] failed to reveal the existence of any spectrum lines[26] under conditions for which the supposed J-rays should appear. It thus became evident that the softening of the secondary X-rays from the lighter elements was due to a different kind of process than the softening of the secondary rays from heavy elements where fluorescent X-rays are present.

A series of skilfully devised absorption experiments performed by J. A. Gray[a] showed, on the other hand, that both in the case of γ-rays and in that of X-rays an increase in wavelength accompanies the scattering of the rays of light elements.

It was at this stage that the first spectroscopic investigations of the secondary X-rays from light elements were made.[b] According to the usual electron theory of scattering it is obvious that the scattered rays will be of the same frequency as the forced oscillations of the electrons which emit them, and hence will be identical in frequency with the primary waves which set the electrons in motion. Instead of showing scattered rays of the same wavelength as the primary rays, however, these spectra revealed lines in the secondary rays corresponding to those in the primary beam, but with each line displaced slightly toward the longer wavelengths.

This result might have been predicted from Gray's absorption measurements; but the spectrum measurements had the advantage of affording a quantitative measurement of the change in wavelength, which gave a basis for its theoretical interpretation.

The spectroscopic experiments which have shown this change in wavelength are too well known[c] to require discussion. The interpretation of the wavelength change in terms of photons being deflected by individual[27] electrons and imparting a part of their energy to the scattering electrons is also very familiar. For purposes of discussion, however, let us recall that when we consider the interaction of a single photon with a single electron the principles of the conservation of energy and momentum lead us[d] to the result that the change in wavelength of the

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[a] E.g., [W.] Duane and [T.] Shimizu, *Phys. Rev.*, **13** (1919), [289]; *ibid.*, **14** (1919), 389.

[a] J. A. Gray, *Phil. Mag.*, **26** (1913), 611; *Jour. Frank. Inst.*, [**190**], 643 (Nov. 1920).

[b] A. H. Compton, *Bull. Natl. Res. Coun.*, No. 20, [18] ([October] 1922); *Phys. Rev.*, **22** (1923), 409.

[c] Cf., e.g., A. H. Compton, *Phys. Rev.*, **22** (1923), 409; P. A. Ross, *Proc. Nat. Acad.*, **10** (1924), 304.

[d] A. H. Compton, *Phys. Rev.*, [**21**] (1923), 483; P. Debye, *Phys. Zeits.*, **24** (1923), 161.

deflected photon is

$$\delta\lambda = \frac{h}{mc}(1 - \cos\varphi) \ , \tag{3}$$

where φ is the angle through which the photon is deflected. The electron at the same time recoils from the photon at an angle of θ given by,[28]

$$\cot\theta = -(1+\alpha)\tan\frac{1}{2}\varphi \ ; \tag{4}$$

and the kinetic energy of the recoiling electron is,

$$E_{\text{kin}} = h\nu\frac{2\alpha\cos^2\theta}{(1+\alpha)^2 - \alpha^2\cos^2\theta} \ . \tag{5}$$

The experiments show in the spectrum of the scattered rays two lines corresponding to each line of the primary ray. One of these lines is of precisely the same wavelength as the primary ray, and the second line, though somewhat broadened, has its centre of gravity displaced by the amount predicted by equation (3). According to experiments by Kallman and Mark[a] and by Sharp,[b] this agreement between the theoretical[29] and the observed shift is precise within a small fraction of 1 per cent.

The recoil electrons. — From the quantitative agreement between the theoretical and the observed wavelengths of the scattered rays, the recoil electrons predicted by the photon theory of scattering were looked for with some confidence.[30] When this theory was proposed, there was no direct evidence for the existence of such electrons, though indirect evidence suggested that the secondary β-rays ejected from matter by hard γ-rays are mostly of this type. Within a few months of their prediction, however, C. T. R. Wilson[c] and W. Bothe[d] independently announced their discovery. The recoil electrons show as short tracks, pointed in the direction of the primary X-ray beam, mixed among the much longer tracks due to the photoelectrons ejected by the X-rays.

Perhaps the most convincing reason for associating these short tracks with the scattered X-rays comes from a study of their number. Each photoelectron in a cloud photograph represents a quantum of truly absorbed X-ray energy. If the short tracks are due to recoil electrons, each one should represent the scattering of a photon. Thus the ratio N_r/N_p of the number of short tracks to the number of long tracks should

[a] H. Kallman and H. Mark, *Naturwiss.*, **13** (1925), 297.
[b] H. M. Sharp, *Phys. Rev.*, **26** (1925), 691.
[c] C. T. R. Wilson, *Proc. Roy. Soc.* [*A*], **104** (1923), 1.
[d] W. Bothe, *Zeits. f. Phys.*, **16** (1923), 319.

be the same as the ratio σ/τ of the scattered to the truly absorbed energy[31] when the X-rays pass through air. The latter ratio is known from absorption measurements, and the former ratio can be determined by counting the tracks on the photographs. The satisfactory agreement between the two ratios[a] for X-rays of different wavelengths means that on the average there is about one quantum of energy scattered for each short track that is produced.

This result is in itself contrary to the predictions of the classical wave theory, since on this basis all the energy spent on a free electron (except the insignificant effect of radiation pressure) should reappear as scattered X-rays. In these experiments, on the contrary, 5 or 10 per cent as much energy appears in the motion of the recoil electrons as appears in the scattered X-rays.

That these short tracks associated with the scattered X-rays correspond to the recoil electrons predicted by the photon theory of scattering becomes clear from a study of their energies. The energy of the electron which produces a track can be calculated from the range of the track. The ranges of tracks which start in different directions have been studied[a] using primary X-rays of different wavelengths, with the result that equation (5)[33] has been satisfactorily verified.

In view of the fact that electrons of this type were unknown at the time the photon theory of scattering was presented, their existence, and the close agreement with the predictions as to their number, direction and velocity, supply strong evidence in favour of the fundamental hypotheses of the theory.

Interpretation of these experiments. — It is impossible to account for scattered rays of altered frequency, and for the existence of the recoil electrons, if we assume that X-rays consist of electromagnetic waves in the usual sense. Yet some progress has been made on the basis of semi-classical theories. It is an interesting fact that the wavelength of the scattered ray according to equation (3)[34] varies with the angle just as one would expect from a Doppler effect if the rays are scattered from an electron moving in the direction of the primary beam. Moreover, the velocity that must be assigned to the electron in order to give the proper magnitude to the change of wavelength is that which the electron would acquire by radiation pressure if it should absorb a quantum of the

[a] A. H. Compton and A. W. Simon, *Phys. Rev.*, **25** (1925), 306; J. M. Nuttall and E. J. Williams, *Manchester Memoirs*, **70** (1926), 1.

[a] Compton and Simon, *loc. cit.*[32]

incident rays. Several writers[a] have therefore assumed that an electron takes from the incident beam a whole quantum of the incident radiation, and then emits this energy as a spherical wave while moving forward[36] with high velocity.

This conception that the radiation occurs in spherical waves, and that the scattering electron can nevertheless acquire suddenly the impulses from a whole quantum of incident radiation is inconsistent with the principle of energy conservation. But there is the more serious experimental difficulty that this theory predicts recoil electrons all moving in the same direction and with the same velocity. The experiments show, on the other hand, a variety of directions and velocities, with the velocity and direction correlated as demanded by the photon hypothesis. Moreover, the maximum range of the recoil electrons, though in agreement with the predictions of the photon theory, is found to be about four times as great as that predicted by the semi-classical theory.

There is nothing in these experiments, as far as we have described them, which is inconsistent with the idea of virtual oscillators continually scattering virtual radiation. In order to account for the change of wavelength on this view, Bohr, Kramers and Slater assumed that the virtual oscillators scatter as if moving in the direction of the primary beam, accounting for the change of wavelength as a Doppler effect. They then supposed that occasionally an electron, under the stimulation of the primary virtual rays, will suddenly move forward with a momentum large compared with the impulse received from the radiation pressure. Though we have seen that not all of the recoil electrons move directly forward, but in a variety of different directions, the theory could easily be extended to include the type of motion that is actually observed.

The only objection that one can raise against this virtual radiation theory in connection with the scattering phenomena as viewed on a large scale, is that it is difficult to see how such a theory could by itself predict the change of wavelength and the motion of the recoil electrons. These phenomena are directly predictable if the conservation of energy and momentum are assumed to apply to the individual actions of radiation on electrons; but this is precisely where the virtual radiation theory denies the validity of the conservation principles.

We may conclude that the photon theory predicts quantitatively and in detail the change of wavelength of the scattered X-rays and the charac-

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[a] C. R. Bauer, *C. R.*, **177** (1923), 1211; C. T. R. Wilson, *Proc. Roy. Soc.* [*A*], **104** (1923), 1; K. Fosterling, *Phys. Zeits.*, **25** (1924), 313; O. Halpern, *Zeits. f. Phys.*, **30** (1924), 153.[35]

teristics of the recoil electrons. The virtual radiation theory is probably not inconsistent with these experiments, but is incapable of predicting the results. The classical theory, however, is altogether helpless to deal with these phenomena.

The origin of the unmodified line — The unmodified line is probably due to X-rays which are scattered by electrons so firmly held within the atom that they are not ejected by the impulse from the deflected photons. This view is adequate to account for the major characteristics of the unmodified rays, though as yet no quantitatively satisfactory theory of their origin has been published.[a] It is probable that a detailed account of these rays will involve definite assumptions regarding the nature and the duration of the interaction between a photon and an electron; but it is doubtful whether such investigations will add new evidence as to the existence of the photons themselves.

A similar situation holds regarding the intensity of the scattered X-rays. Historically it was the fact that the classical electromagnetic theory is unable to account for the low intensity of the scattered X-rays which called attention to the importance of the problem of scattering. But the solutions which have been offered by Breit,[b] Dirac[c] and others[d] of this intensity problem as distinguished from that of the change of wavelength, seem to introduce no new concepts regarding the nature of radiation or of the scattering process. Let us therefore turn our attention to the experiments that have been performed on the individual process of interaction between photons and electrons.

INTERACTIONS BETWEEN RADIATION AND SINGLE ELECTRONS[39]

The most significant of the experiments which show departures from the predictions of the classical wave theory are those that study the action of radiation on individual atoms or on individual electrons. Two

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[a] Cf., however, G. E. M. Jauncey, *Phys. Rev.*, **25** (1925), 314 and *ibid.*, 723; G. Wentzel, *Zeits. f. Phys.*, **43** (1927), 14, 779; I. Waller, *Nature*, [**120**, 155] (July 30, 1927).[37][The footnote in the English edition continues with the sentence: 'It is possible that the theories of the latter authors may be satisfactory, but they have not yet been stated in a form suitable for quantitative test' (*eds.*).]

[b] G. Breit, *Phys. Rev.*, **27** (1926), 242.

[c] P. A. M. Dirac, *Proc. Roy. Soc. A*, [**111**] (1926), [405].

[d] W. Gordon, *Zeits. f. Phys.*, **40** (1926), 117; E. Schrödinger, *Ann. d. Phys.*, **82** (1927), 257; O. Klein, *Zeits. f. Phys.*, **41** (1927), 407; G. Wentzel, *Zeits. f. Phys.*, **43** (1927), 1, 779.[38]

methods have been found suitable for performing these experiments, Geiger's point counters, and Wilson's cloud expansion photographs.

(1) *Test for coincidences with fluorescent X-rays.* — Bothe has performed an experiment[a] in which fluorescent K radiation from a thin copper foil is excited by a beam of incident X-rays. The emitted rays are so feeble that only about five quanta of energy are radiated per second. Two point counters are mounted, one on either side of the copper foil in each of which an average of one photoelectron is produced and recorded for about twenty quanta radiated by the foil. If we assume that the fluorescent radiation is emitted in quanta of energy, but proceed[s] in spherical waves in all directions, there should thus be about 1 chance in 20 that the recording of a photoelectron in one chamber should be simultaneous with the recording of a photoelectron in the other.

The experiments showed no coincidences other than those which were explicable by such sources as high-speed β-particles which traverse both counting chambers.

This result is in accord with the photon hypothesis,[b] according to which coincidences should not occur. It is, nevertheless, equally in accord with the virtual radiation hypothesis, if one assumes that the virtual oscillators in the copper continuously emit virtual fluorescent radiation, so that the photoelectrons should be observed in the counting chambers at arbitrary intervals.[c]

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[a] W. Bothe, *Zeits. f. Phys.*, **37** (1926), 547.

[b] The English edition continues: 'For if a photon of fluorescent radiation produces a β-ray in one counting chamber it cannot traverse the second chamber. Coincidences should therefore not occur' (*eds.*).

[c] At this point in the English version Compton is much more critical of the BKS theory (*eds.*):

According to the virtual radiation hypothesis, however, coincidences should have been observed. For on this view the fluorescent K radiation is emitted by virtual oscillators associated with atoms in which there is a vacancy in the K shell. That is, the copper foil can emit fluorescent K radiation only during the short interval of time following the expulsion of a photoelectron from the K shell, until the shell is again occupied by another electron. This time interval is so short (of the order of 10^{-15} sec.) as to be sensibly instantaneous on the scale of Bothe's experiments. Since on this view the virtual fluorescent radiation is emitted in spherical waves, the counting chambers on both sides of the foil should be simultaneously affected, and coincident pulses in the two chambers should frequently occur. The results of the experiment are thus contrary to the predictions of the virtual radiation hypothesis.

But the experiment is important in the sense that it refutes the often suggested idea that a quantum of radiation energy is suddenly emitted in the form of a spherical wave when an atom passes from one stationary state to another.

(2) *The composite photoelectric effect.*[40] — Wilson[a] and Auger[b] have noticed in their cloud expansion photographs that when X-rays eject photoelectrons from heavy atoms, it often occurs that two or more electrons are ejected simultaneously from the same atom. Auger has deduced from studying the ranges of these electrons that, when this occurs, the total energy of all the emitted electrons is no larger than that of a quantum of the incident radiation. When two electrons are emitted simultaneously it is usually the case that the enegy of one of them is

$$E_{\mathrm{kin}} = h\nu - h\nu_K \ ,$$

which according to the photon theory means that this electron is due to the absorption of an incident photon accompanied by the ejection of an electron from the K energy level. The second electron has in general the energy

$$E_{\mathrm{kin}} = h\nu_K - h\nu_L \ .$$

This electron can be explained as the result of the absorption by an L electron of the Kα-ray emitted when another L electron occupies the place left vacant in the K orbit by the primary photoelectron. It is established that all the electrons that are observed in the composite photoelectric effect have to be interpreted in the same way. Their interpretation according to the photon theory thus meets with no difficulties.

With regard to the virtual radiation theory, we can take two points of view: first, under the influence of the excitation produced by the primary virtual radiation, virtual fluorescent K radiation is emitted by virtual oscillators associated with all the atoms traversed by the primary beam. In this view, the probability that this virtual fluorescent radiation will cause the ejection of a photoelectron from the same atom as the one that has emitted the primary photoelectron is so small that such an event will almost never occur; second, we can alternatively assume that a virtual oscillator emitting virtual K radiation is associated only with an atom in which there is a vacant place in the K shell. In this case,

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[a] C. T. R. Wilson, *Proc. Roy. Soc. A*, **104** (1923), 1.
[b] P. Auger, *Journ. d. Phys.*, **6** (1926), 183.

since the virtual radiation proceeds from the atom that has emitted the primary photoelectron, we could expect with extremely large probability that it should excite a photoelectron from the L shell of its own atom, thus accounting for the composite photoelectric effect. But in this view the virtual fluorescent radiation is emitted only during a very short interval after the ejection of the primary photoelectron, in which case Bothe's fluorescence experiment, described above, should have shown some coincidences.

One sees thus that the virtual radiation hypothesis is irreconcilable both with the composite photoelectric effect and with the absence of coincidences in Bothe's fluorescence experiment. The photon hypothesis, instead, is in complete accord with both these experimental facts.

(3) *Bothe and Geiger's coincidence experiments.*[41] — We have seen that according to Bohr, Kramers and Slater's theory, virtual radiation[42] is being continually scattered by matter traversed by X-rays, but only occasionally is a recoil electron emitted. This is in sharp contrast with the photon theory, according to which a recoil electron appears every time a photon is scattered. A crucial test between the two points of view is afforded by an experiment devised and brilliantly performed[43] by Bothe and Geiger.[a] X-rays were passed through hydrogen gas, and the resulting recoil electrons and scattered rays were detected by means of two different point counters placed on opposite sides of the column of gas. The chamber for counting the recoil electrons was left open, but a sheet of thin platinum prevented the recoil electrons from entering the chamber for counting the scattered rays. Of course not every photon entering the second counter could be noticed, for its detection depends upon the production of a β-ray. It was found that there were about ten recoil electrons for every scattered photon that recorded itself.

The impulses from the counting chambers were recorded on a moving photographic film. In observations over a total period of over five hours, sixty-six such coincidences were observed. Bothe and Geiger calculate that according to the statistics of the virtual radiation theory the chance was only 1 in 400 000 that so many coincidences should have occurred. This result therefore is in accord with the predictions of the photon theory, but is directly contrary to the statistical view of the scattering process.

(4) *Directional emission of scattered X-rays.* — Additional information

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[a] W. Bothe and H. Geiger, *Zeits. f. Phys.*, **26** (1924), 44; **32** (1925), 639.

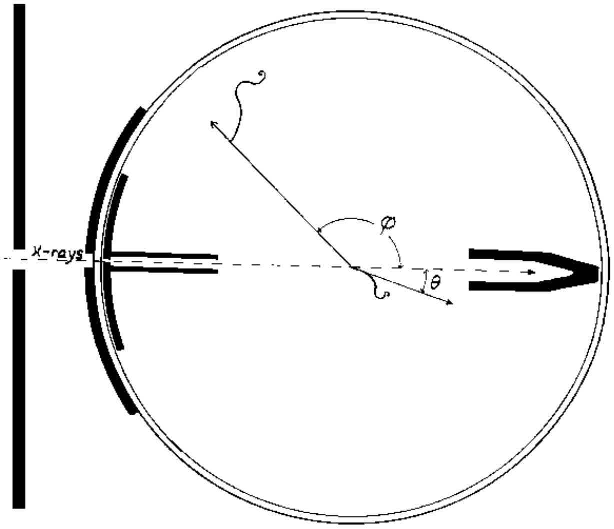

Fig. 3. If the X-rays excite a recoil electron at an angle θ, the photon theory predicts a secondary β-particle at an angle φ.

regarding the nature of scattered X-rays has been obtained by studying the relation between the direction of ejection of the recoil electron and the direction in which the associated photon proceeds. According to the photon theory, we have a definite relation (equation (4)) between the angle at which the photon is scattered and the angle at which the recoil electron is ejected. But according to any form of spreading wave theory, including that of Bohr, Kramers and Slater, the scattered rays may produce effects in any direction whatever, and there should be no correlation between the directions in which the recoil electrons proceed and the directions in which the secondary β-rays are ejected by the scattered X-rays.

A test to see whether such a relation exists has been made,[a] using Wilson's cloud apparatus, in the manner shown diagrammatically in Fig. 3. Each recoil electron produces a visible track, and occasionally a secondary track is produced by the scattered X-ray. When but one recoil

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[a] A. H. Compton and A. W. Simon, *Phys. Rev.*, **26** (1925), 289.

electron appears on the same plate with the track due to the scattered rays, it is possible to tell at once whether the angles satisfy equation (4). If two or three recoil tracks appear,[44] the measurements on each track can be approximately[45] weighted.

Out of 850 plates taken in the final series of readings, thirty-eight show both recoil tracks and secondary β-ray tracks. On eighteen of these plates the observed angle φ[46] is within 20 degrees of the angle calculated from the measured value of θ, while the other twenty tracks are distributed at random angles. This ratio 18:20 is about that to be expected for the ratio of the rays scattered by the part of the air from which the recoil tracks could be measured to the stray rays from various sources. There is only about 1 chance in 250 that so many secondary β-rays should have appeared at the theoretical angle.

If this experiment is reliable, it means that there is scattered X-ray energy associated with each recoil electron sufficient to produce a β-ray, and proceeding in a direction determined at the moment of ejection of the recoil electron. In other words, the scattered X-rays proceed in photons, that is[47] in directed quanta of radiant energy.

This result, like that of Bothe and Geiger, is irreconcilable with Bohr, Kramers and Slater's hypothesis of the statistical production of recoil and photoelectrons. On the other hand, both of these experiments are in complete accord with the predictions of the photon theory.

Reliability of experimental evidence

While all of the experiments that we have considered are difficult to reconcile with the classical theory that radiation consists of electromagnetic waves, only those dealing with the individual scattering process[48] afford crucial tests between the photon theory and the statistical theory of virtual radiation. It becomes of especial importance, therefore, to consider the errors to which these experiments are subject.

When two point counters are set side by side, it is very easy to obtain coincidences from extraneous sources. Thus, for example, the apparatus must be electrically shielded so perfectly that a spark on the high-tension outfit that operates the X-ray tube may not produce coincident impulses in the two counters. Then there are high-speed α- and β-rays, due to radium emanation in the air and other radioactive impurities, which may pass through both chambers and produce spurious coincidences. The method which Bothe and Geiger used to detect the coincidences, of[49] recording on a photographic film the time of each pulse, makes it possible to estimate reliably[50] the probability that the coincidences are due to chance. Moreover, it is possible by auxiliary tests to determine whether spurious coincidences are occurring — for example, by operating the outfit as usual, except that the X-rays are absorbed by a sheet of lead. It is especially worthy of note that in the fluorescence experiment the photon theory predicted absence of coincidences, while in the scattering experiment it predicted their presence. It is thus difficult to see how both of these counter experiments can have been seriously affected by systematic errors.

In the cloud expansion experiment the effect of stray radiation is to hide the effect sought for, rather than to introduce a spurious effect. It is possible that due to radioactive contamination and to stray scattered X-rays β-particles may appear in different parts of the chamber, but it will be only a matter of chance if these β-particles appear in the position predicted from the direction of ejection of the recoil electrons. It was in fact only by taking great care to reduce such stray radiations to a minimum that the directional relations were clearly observed in the photographs. It would seem that the only form of consistent error that could vitiate the result of this experiment would be the psychological[51] one of misjudging the angles at which the β-particles appear. It hardly seems possible, however, that errors in the measurement of these angles could be large enough to account for the strong apparent tendency for the angles to fit with the theoretical formula.

It is perhaps worth mentioning further that the initial publications of the two experiments on the individual scattering process were made simultaneously, which means that both sets of experimenters had independently reached a conclusion opposed to the statistical theory of the production of the β-rays.

Nevertheless,[52] given the difficulty of the experiments and the importance of the conclusions to which they have led, it is highly desirable that both experiments should be repeated by physicists from other laboratories.

SUMMARY

The classical theory that radiation consists of electromagnetic waves propagated in all directions through space[53] is intimately connected to the idea of the ether, which is difficult to conceive. It affords no adequate picture of the manner in which radiation is emitted or absorbed. It is inconsistent with the experiments on the photoelectric effect, and is entirely helpless to account for the change of wavelength of scattered radiation or the production of recoil electrons.

The theory of virtual oscillators and virtual radiation which are associated statistically with sudden jumps of atomic energy and the emission of photoelectrons and recoil electrons, does not seem to be inconsistent with any of these phenomena as viewed on a macroscopic scale. This theory, however,[54] retains the difficulties inherent in the conception of the ether and seems powerless to predict the characteristics of the photoelectrons and the recoil electrons. It[55] is further difficult to reconcile with the composite photoelectric effect and is also contrary to Bothe's and Bothe and Geiger's coincidence experiments and to the ray track experiments relating the directions of ejection of a recoil electron and of emission of the associated scattered X-ray.

The photon theory avoids the difficulties associated with the conception of the ether.[56] The production and absorption of radiation is very simply connected with the modern idea of stationary states. It supplies a straightforward explanation of the major characteristics of the photoelectric effect, and it accounts in the simplest possible manner for the change of wavelength accompanying scattering and the existence of recoil electrons. Moreover, it predicts accurately the results of the experiments with individual radiation quanta, where the statistical theory fails.

Unless the four[57] experiments on the individual events[58] are subject to improbably large experimental errors, the conclusion is, I believe, unescapable that radiation consists of directed quanta of energy, i.e., of photons, and that energy and momentum are conserved when these photons interact with electrons or atoms.

Let me say again that this result does not mean that there is no truth in the concept of waves of radiation. The conclusion is rather that energy is not transmitted by such waves. The power of the wave concept in problems of interference, refraction, etc., is too well known to require emphasis. Whether the waves serve to guide the photons, or whether there is some other relation between photons and waves is another and a difficult question.

Discussion of Mr Compton's report

Mr Lorentz. — I would like to make two comments. First on the question of the ether. Mr Compton considers it an advantage of the photon theory that it allows us to do without the hypothesis of an ether which leads to great difficulties. I must say that these difficulties do not seem so great to me and that in my opinion the theory of relativity does not necessarily rule out the concept of a universal medium. Indeed, Maxwell's equations are compatible with relativity, and one can well imagine a medium for which these equations hold. One can even, as Maxwell and other physicists have done with some success, construct a mechanical model of such a medium. One would have to add only the hypothesis of the permeability of ponderable matter by the ether to have all that is required. Of course, in making these remarks, I should not wish to return in any way to these mechanical models, from which physics has turned away for good reasons. One can be satisfied with the concept of a medium that can pass freely through matter and to which Maxwell's equations can be applied.

In the second place: it is quite certain that, in the phenomena of light, there must yet be something other than the photons. For instqncem in a diffraction experiment performed with very weak light, it can happen that the number of photons present at a given instant between the diffracting screen and the plane on which one observes the distribution of light, is very limited. The average number can even be smaller than one, which means that there are instants when no photon is present in the space under consideration.

This clearly shows that the diffraction phenomena cannot be produced by some novel action among the photons. There must be something that guides them in their progress and it is natural to seek this something in the electromagnetic field as determined by the classical theory. This notion of electromagnetic field, with its waves and vibrations would bring us back, in Mr Compton's view, to the notion of ether.

Mr Compton. — It seems, indeed, difficult to avoid the idea of waves in the discussion of optical phenomena. According to Maxwell's theory the electric and magnetic properties of space lead to the idea of waves as directly as did the elastic ether imagined by Fresnel. Why the space having such magnetic properties should bear the name of ether is perhaps simply a matter of words. The fact that these properties of space immediately lead to the wave equation with velocity c is a much more solid basis for the hypothesis of the existence of waves than the old elastic ether. That *something* (E and H) propagates like a wave with velocity c seems evident. However, experiments of the kind we have just discussed show, if they are correct, that the *energy* of the bundle of X-rays propagates in the form of *particles* and not in the form of extended waves. So then, not even the electromagnetic ether appears to be satisfactory.

MR BRAGG. — In his report Mr Compton has discussed the average momentum component of the electrons in the direction of motion of the photon, and he has informed us of the conclusion, at which several experimenters have arrived, that this forward average component is equal to the momentum of the light quantum whose energy has been absorbed and is found again in that of the photoelectron.

I would like to report in this connection some results obtained by Mr Williams.[a] Monochromatic X-rays, with wavelength lying between 0.5 Å and 0.7 Å, enter a Wilson cloud chamber containing oxygen or nitrogen. The trajectories of the photoelectrons are observed through a stereoscope and their initial directions are measured. Since the speed of the photoelectrons is exactly known (the ionisation energy being weak by comparison to the quantity $h\nu$), a measurement of the initial direction is equivalent to a measurement of momentum in the forward direction. Williams finds that the average momentum component in this direction is in all cases markedly larger than the quantity $\frac{h\nu}{c}$ or $\frac{h}{\lambda}$. These results can be summarised by a comparison with the scheme proposed by Perrin and Auger ([P. Auger and F. Perrin], *Journ. d. Phys.* [6th series, vol. **8**] (February 1927), [93]). They are in perfect agreement with the $\cos^2\theta$ law, provided one assumes that the magnetic impulse T_m is equal to $1.8\frac{h\nu}{c}$ and not just $\frac{h\nu}{c}$ as these authors assume. One should not attach any particular importance to this number 1.8, because the range of the examined wavelengths is too small. I mention it only to show that it is possible that the simple law proposed by Mr Compton might not be exact.

I would like to point out that this method of measuring the forward component of the momentum is more precise than an attempt made to establish results about the most probable direction of emission.

MR WILSON says that his own observations, discussed in his Memoir

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[a] Cf. the relevant footnote on p. 340 (*eds.*).

of 1923[a] (but which do not pretend to be very precise) seem to show that in fact the forward momentum component of the photoelectrons is, on average, much larger than what one would derive from the idea that the absorbed quantum yields all of its momentum to the expelled electron.

Mr Richardson. — When they are expelled by certain X-rays, the electrons have a momentum in the direction of propagation of the rays equal to $1.8\frac{h\nu}{c}$. If I have understood Mr Bragg correctly, this result is not the effect of some specific elementary process [action], but the average result for a great number of observations in which the electrons were expelled in different directions. Whether or not the laws of energy and momentum conservation apply to an elementary process, it is certain that they apply to the average result for a great number of these processes. Therefore, the process [processus] we are talking about must be governed by the equations for momentum and energy. If for simplicity we ignore the refinements introduced by relativity, these equations are

$$\frac{h\nu}{c} = m\overline{v} + M\overline{V}$$

and

$$h\nu = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}M\overline{V}^2 \ ,$$

where m and M are the masses, v and V the velocities of the electrons and of the positive residue; the overbars express that these are averages. The experiments show that the average value of mv is $1.8\frac{h\nu}{c}$ and not $\frac{h\nu}{c}$. This means that $M\overline{V}$ is not zero, so that we cannot ignore this term in the equation. If we consider, for instance, the photoelectric effect on a hydrogen atom, we have to take the collision energy of the hydrogen nucleus into account in the energy equation.

Mr Lorentz. — The term $\frac{1}{2}M\overline{V}^2$ will however be much smaller than $\frac{1}{2}m\overline{v}^2$?[59]

Mr Richardson. — It is approximately its 1850th part: that cannot always be considered negligible.

Mr Born thinks that he is speaking also for several other members

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[a] Referenced in footnote on p. 336 (*eds.*).

in asking Mr Compton to explain why one should expect that the momentum imparted to the electron be equal to $\frac{h\nu}{c}$.

Mr Compton. — When radiation of energy $h\nu$ is absorbed by an atom — which one surely has to assume in order to account for the kinetic energy of the photoelectron — the momentum imparted to the atom by this radiation is $\frac{h\nu}{c}$. According to the classical electron theory, when an atom composed of a negative charge $-e$ of mass m and a positive charge $+e$ of mass M absorbs energy from an electromagnetic wave, the momenta imparted to the two elementary charges [électrons] are inversely proportional to their masses. This depends on the fact that the forward momentum is due to the magnetic vector, which acts with a force proportional to the velocity and consequently more strongly on the charge having the smaller mass.[60] Effectively, the momentum is thus received by the charge with the smaller mass.

Mr Debye. — Is the reason why you think that the rest of the atom does not receive any of the forward momentum purely theoretical?

Mr Compton. — The photographs of the trajectories of the photoelectrons show, in accordance with Auger's prediction, that the forward component of the momentum of the photoelectron is, on average, the same as that of the photon. That means, clearly, that on average the rest of the atom does not receive any momentum.

Mr Dirac. — I have examined the motion of an electron placed in an arbitrary force field according to the classical theory, when it is subject to incident radiation, and I have shown in a completely general way that at every instant the fraction of the rate of change [vitesse de variation] of the forward momentum of the electron due to the incident radiation is equal to $\frac{1}{c}$ times the fraction of the rate of change of the energy due to the incident radiation. The nucleus and the other electrons of the atom produce changes of momentum and of energy that at each instant are simply added to those produced by the incident radiation. Since the radiation must modify the electron's orbit, it must also change the fraction of the rate of change of the momentum and of the energy that comes from the nucleus and the other electrons, so that it would be necessary to integrate the motion in order to determine the total change produced by the incident radiation in the energy and the momentum.

Mr Born. — I would like to mention here a paper by Wentzel,[a] which contains a rigorous treatment of the scattering of light by atoms according to quantum mechanics. In it, the author considers also the influence of the magnetic force, which allows him to obtain the quantum analogue of the classical light pressure. It is only in the limiting case of very short wavelengths that one finds that the momentum of light $\frac{h\nu}{c}$ is completely transmitted to the electron; in the case of large wavelengths an influence of the binding forces appears.

Mr Ehrenfest. — One can show by a very simple example where the surplus of forward momentum, which we have just discussed, can have its origin. Take a box whose inner walls reflect light completely, but diffusedly, and assume that on the bottom there is a little hole. Through the latter I shine a ray of light into the box which comes and goes inside the box and pushes away its lid and bottom. The lid then has a surplus of forward momentum.

Mr Bohr.[a] — With regard to the question of waves or photons discussed by Mr Compton, I would like to make a few remarks, without preempting the general discussion. The radiation experiments have indeed revealed features that are not easy to reconcile within a classical picture. This difficulty arises particularly in the Compton effect itself. Several aspects of this phenomenon can be described very simply with the aid of photons, but we must not forget that the change of frequency that takes place is measured using instruments whose functioning is interpreted according to the wave theory. There seems to be a logical contradiction here, since the description of the incident wave as well as that of the scattered wave require that these waves be finitely extended [limitées] in space and time, while the change in energy and in momentum of the electron is considered as an instantaneous phenomenon at a given point in spacetime. It is precisely because of such difficulties that Messrs Kramers, Slater and myself were led to think that one should completely reject the idea of the existence of photons and assume that the laws of conservation of energy and momentum are true only in a statistical way.

The well-known experiments by Geiger and Bothe and by Compton and Simon, however, have shown that this point of view is not admissible

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[a] Born is presumably referring to Wentzel's second paper on the photoelectric effect (Wentzel 1927). Compare Mehra and Rechenberg (1987, pp. 835 ff.) (*eds.*).

[a] This discussion contribution by Bohr is reprinted and translated also in vol. 5 of Bohr's *Collected Works* (Bohr 1984, pp. 207–12). (*eds.*).

and that the conservation laws are valid for the individual processes, in accordance with the concept of photons. But the dilemma before which we are placed regarding the nature of light is only a typical example of the difficulties that one encounters when one wishes to interpret the atomic phenomena using classical concepts. The logical difficulties with a description in space and time have since been removed in large part by the fact that it has been realised that one encounters a similar paradox with respect to the nature of material particles. According to the fundamental ideas of Mr de Broglie, which have found such perfect confirmation in the experiments of Davisson and Germer, the concept of waves is as indispensable in the interpretation of the properties of material particles as in the case of light. We know thereby that it is equally necessary to attribute to the wave field a finite extension in space and in time, if one wishes to define the energy and the momentum of the electron, just as one has to assume a similar finite extension in the case of the light quantum in order to be able to talk about frequency and wavelength.

Therefore, in the case of the scattering process, in order to describe the two changes affecting the electron and the light we must work with four wave fields (two for the electron, before and after the phenomenon, and two for the quantum of light, incident and scattered), finite in extension, which meet in the same region of spacetime.[a] In such a representation all possibility of incompatibility with a description in space and time disappears. I hope the general discussion will give me the opportunity to enter more deeply into the details of this question, which is intimately tied to the general problem of quantum theory.

Mr Brillouin. — I have had the opportunity to discuss Mr Compton's report with Mr Auger,[b] and wish to make a few comments on this topic. A purely corpuscular description of radiation is not sufficient to understand the peculiarities of the phenomena; to assume that energy is transported by photons $h\nu$ is not enough to account for all the effects of radiation. It is essential to complete our information by giving the direction of the electric field; we cannot do without this field, whose role in the wave description is well known.

I shall recall in this context a simple argument, recently given by Auger

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[a] Compare also the discussion contributions below, by Pauli, Schrödinger and others (*eds.*).

[b] As noted in section 1.4, this and other reports had been circulated among the participants before the conference (*eds.*).

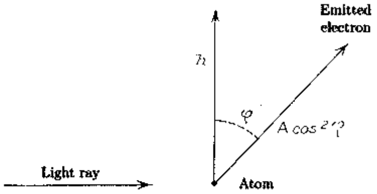

Fig. 1.

and F. Perrin, and which illustrates clearly this remark. Let us consider the emission of electrons by an atom subject to radiation, and let us examine the distribution of the directions of emission. This distribution has usually been observed in a plane containing the light ray and the direction of the electric field (the incident radiation is assumed to be polarised); let φ be the angle formed by the direction of emission of the photoelectrons and the electric field h; as long as the incident radiation is not too hard, the distribution of the photoelectrons is symmetric around the electric field; one can then show that the probability law necessarily takes the form $A\cos^2\varphi$. Indeed, instead of observing the distribution in the plane of incidence (Fig. 1), let us examine it in the plane of the wave; the same distribution law will still be valid; and it is the only one that would allow us to obtain, through the superposition of two waves polarised at right angles, an entirely symmetric distribution

$$A\cos^2\varphi + A\sin^2\varphi = A\ .$$

Now, from the point of view of waves, one must necessarily obtain this result, a beam [rayonnement] of natural light having no privileged direction in the plane of the wave. These symmetry considerations, which any theory of radiation must respect, provide a substantial difficulty for the structural theories of the photon (Bubb's quantum vector, for instance).

Summing up, the discontinuity of the radiation manifests itself just in the most elementary way, through the laws of conservation of energy and momentum, but the detailed analysis of the phenomena is interpreted more naturally from the continuous point of view. For the problem of emission of the photoelectrons, a complete theory has been given by

Wentzel, by means of wave mechanics.[a] He finds the $A\cos^2\varphi$ law of F. Perrin and Auger for radiation of low penetration; when the radiation is harder, Wentzel obtains a more complex law, in which the electrons tend to be emitted in larger numbers in the forward direction. His theory, however, seems incomplete with regard to this point since, if I am not mistaken, he has assumed the immobility of the atomic nucleus; now, nothing tells us *a priori* how the momentum $\frac{h\nu}{c}$ of the photon is going to be distributed between the nucleus and the emitted electron.

Mr Lorentz. — Allow me to point out that according to the old electron theory, when one has a nucleus and an electron on which a beam of polarised light falls, the initial angular momentum of the system is always conserved. The angular momentum imparted to the electron-nucleus system will be provided at the expense of the angular momentum of the radiation field.

Mr Compton. — The conception of the photon differs from the classical theory in that, when a photoelectron is emitted, the photon is completely absorbed and no radiation field is left. The motion of the photoelectron must thus be such that the final angular momentum of the electron-nucleus system will be the same as the initial momentum of the photon-electron-nucleus system. This condition restricts the possible trajectories of the emitted photoelectron.

Mr Kramers. — In order to interpret his experiments, Mr Compton needs to know how the absorption μ is divided between a component τ, due to the 'true' absorption, and a component σ due to the scattering. We do not know with certainty that, if μ can be written in the form $C\lambda^a + D$, the constant D truly represents the scattering for large wavelengths, where $C\lambda^a$ is no longer small compared to D. In general, thus, specific measurements of σ are necessary. Did you have sufficient information regarding the values of σ and τ in your experiments?

Mr Compton. — The most important case in which it is necessary to distinguish between the true absorption τ and the absorption σ due to the scattering, is that of carbon. For this case, Hewlett[a] has measured

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[a] This is presumably Wentzel's treatment from his first paper on the photoelectric effect (Wentzel 1926). Compare again Mehra and Rechenberg (1987, pp. 835 ff.) (*eds.*).

[a] [C. W. Hewlett, *Phys. Rev.*, **17** (1921), 284.]

σ directly for the wavelength 0.71 Å and the total absorption μ over a large range of wavelengths. The difference between μ and σ for the wavelength 0.71 Å corresponds to τ for this wavelength. According to Owen's formula this τ is proportional to λ^3; we can thus calculate τ for all wavelengths. The difference between this value of τ and the measured value of μ corresponds to the value of σ for the wavelengths considered. Since τ is relatively small in the case of carbon, especially for small wavelengths, this procedure yields a value for σ that cannot be very imprecise.

Mr Bragg. — When one consults the original literature on this subject, one is struck by how much the X-ray absorption measurements leave to be desired, both with regard to precision as well as with regard to the extent of the scale of wavelengths for which they have been performed.

Mr Pauli. — How large is the broadening of the modified rays?

Mr Compton. — The experiments have shown clearly that the modified ray is broader than the unmodified ray. In the typical case of the ray λ 0.7 Å scattered by carbon, the broadening is of order 0.005 angström. Unfortunately, the experiments concerning this point are far from being satisfactory, and this number should be considered only as a rough approximation.

Mr Pauli. — The broadening of the modified ray can be interpreted theoretically in two ways, which to tell the truth reduce to the same according to quantum mechanics. First, the electron, in a given stationary state of the atom, has a certain velocity distribution with regard to magnitude and direction. That gives rise to a broadening of the frequency of the scattered rays through the Doppler effect, a broadening whose order of magnitude is $\frac{\Delta\lambda}{\lambda} = \frac{v}{c}$, where v denotes the average velocity of the electron in the atom.

In order to convey the second means of explanation, I would like to sketch briefly the meaning of the Compton effect in wave mechanics.[a] This meaning is based first of all on the wave equation

$$\sum_\alpha \frac{\partial^2\psi}{\partial x_\alpha^2} - \frac{4\pi i}{h}\frac{e}{c}\sum_\alpha \varphi_\alpha \frac{\partial\psi}{\partial x_\alpha} - \frac{4\pi^2}{h^2}\left(\frac{e^2}{c^2}\sum_\alpha \varphi_\alpha^2 + m_0^2c^2\right)\psi = 0$$

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[a] For a modern discussion, see Björken and Drell (1964, Chapter 9) (*eds.*).

and further on the expression

$$iS_\alpha = \psi \frac{\partial \psi^*}{\partial x_\alpha} - \psi^* \frac{\partial \psi}{\partial x_\alpha} + \frac{4\pi i}{h} \frac{e}{c} \varphi_\alpha \psi \psi^* \ ,$$

in which ψ is Schrödinger's function, ψ^* the complex conjugate value and φ_α the four-potential of the electromagnetic field. Given S_α, one calculates the radiation from *classical* electrodynamics. If now in the wave equation one replaces φ_α by the potential of an incident plane wave, the terms that are proportional to the amplitude of this wave can be considered infinitely small in the first order, and one can apply the approximation methods of perturbation theory. This now is a point where one needs to be especially careful. It is all-important to know what one will take as the unperturbed field ψ, which must correspond to a solution of the wave equation for the free particle (corresponding to $\varphi_\alpha = 0$).[61] One finds that in order to agree with the observations, it is necessary to take *two* infinitely extended monochromatic wave trains as being already present in the unperturbed solution, of which one corresponds to the initial state, the other to the final state of the Compton process. In my opinion this assumption, on which the theories of the Compton effect by Schrödinger, Gordon and Klein are based, is unsatisfactory and this defect is corrected only by Dirac's quantum electrodynamics.[a] But if one makes this assumption, the current distribution of the unperturbed solution corresponds to that of an infinitely extended diffraction grating [un réseau infiniment étendu] that moves with a constant speed, and the action of the radiation on this grating leads to a sharp modified ray.

If one considers a bound electron in an atom, one has to replace one component of the solution ψ in the unperturbed charge and current distribution by the eigenfunction of the atom in the stationary state considered, and the other component by a solution corresponding to the final state of the Compton process (belonging to the continuous spectrum of the atom), which at great distance from the atom behaves more or less as a plane wave. One thus has a moving grating that first of all depends only on the finite extension of the atom and in the second place has components no longer moving with the same speed at all. This gives rise to a lack of sharpness of the shifted ray of the scattered radiation.

But one can show that, from the point of view of quantum mechanics, this explanation for the lack of sharpness of the shifted ray is just another form of the explanation given in the first instance and which relies on the different directions of the initial velocities of the electrons in the atom.

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[a] Compare below Schrödinger's contribution and the ensuing discussion (*eds.*).

For according to quantum mechanics if

$$\psi = f(x, y, z) e^{2\pi i v t}$$

is the eigenfunction corresponding to a given stationary state of the atom, the function[62]

$$\varphi(p_x, p_y, p_z) = \int\int\int f(x, y, z) e^{-\frac{2\pi i}{\lambda}(p_x x + p_y y + p_z z)} dx dy dz \ ,$$

which one obtains by decomposing f in plane waves according to Fourier can be interpreted in the sense that $|\varphi(p)|^2 dp_x dp_y dp_z$ denotes the probability that in the given stationary state the components of the momentum of the electron lie between p_x, p_y, p_z and $p_x + dp_x$, etc. Now, if through the resulting velocity distribution of the electrons in the atom one calculates the broadening of the shifted line according to the first point of view, for light of sufficiently short wavelength with respect to which the electron can be considered free in the atom (and it is only under these conditions that the procedure is legitimate), one finds exactly the same result as with the other method described.[a]

MR COMPTON. — Jauncey has calculated the broadening of the modified ray using essentially the method that Mr Pauli has just described. Jauncey assumed, however, that the velocities of the electron are the ones given by Bohr's theory of orbital motions. The broadening thus obtained is larger than that found experimentally.

MRS CURIE. — In his very interesting report, Professor Compton has dwelt on emphasising the reasons that lead one to adopt the theory of a collision between a quantum and a free electron. Along the same line of thought, I think it is useful to point out the following two views:

First, the existence of collision electrons seems to play a fundamental role in the biological effects produced on living tissues by very high-frequency radiation, such as the most penetrating γ-rays emitted by radioelements. If one assumes that the biological effect may be attributed to the ionisation produced in the cells subjected to radiation, this effect cannot depend directly on the γ-rays, but is due to the emission of secondary β-rays that accompanies the passage of the γ-rays through matter. Before the discovery of the collision electrons, only a single

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[a] Pauli was possibly the first to introduce the probability interpretation of the wave function in momentum space, in a letter to Heisenberg of 19 October 1926 (Pauli, 1979, pp. 347–8). Cf. the footnote on p. 117 (*eds.*).

mechanism was known for the production of these secondary rays, that consisting in the total absorption of a quantum of radiation by the atom, with the emission of a photoelectron. The absorption coefficient τ relating to this process varies with the wavelength λ of the primary γ-radiation, as well as with the density $[\rho]$ of the absorbing matter and the atomic number N of the atoms composing it, according to the well-known relation of Bragg and Peirce $\frac{\tau}{\rho} = \Lambda N^3 \lambda^3$, where Λ is a coefficient that has a constant value for frequencies higher than that of the K discontinuity. If this relation valid in the domain of X-rays can be applied to high-frequency γ-rays, the resulting value of $\frac{\tau}{\rho}$ for the light elements is so weak that the emission of photoelectrons appears unable to explain the biological effects of radiation on the living tissues traversed.[a]

The issue appears altogether different if one takes into consideration the emission of collision electrons in these tissues, following Compton's theory. For a collimated primary beam of γ-rays, the fraction of electromagnetic energy converted into kinetic energy of the electrons per unit mass of the absorbing matter is given by the coefficient

$$\frac{\sigma_a}{\rho} = \frac{\alpha}{(1-2\alpha)^2} \frac{\sigma_0}{\rho} ,$$

where $\frac{\sigma_0}{\rho}$ is the scattering coefficient per unit mass valid for medium frequency X-rays, according to the theory of J. J. Thomson, and is close to 0.2, while α is Compton's parameter $\alpha = \frac{h\nu}{mc^2}$ (h Planck's constant, ν primary frequency, m rest mass of the electron, c speed of light). Taking $\alpha = 1.2$, a value suitable for an important group of γ-rays (equivalent potential 610 kilovolts), one finds $\frac{\sigma_a}{\rho} = 0.02$, that is, 2 per cent of the primary energy is converted to energy of the electron per unit mass of absorbing matter, whence a possibility of interpreting the observed biological effects. To this direct production of collision electrons along the trajectory of the primary beam is added, in an extended medium, a supplementary production, from the fact that to each of these electrons corresponds a scattered quantum, with a smaller value than the primary quantum, and that this scattered quantum can in turn be subject to the Compton effect in the medium through which it propagates, with production of a new collision electron and of an even smaller quantum. This process, indefinitely repeatable and called the 'multiple Compton effect' seems in fact to have been observed by certain authors.[a] Not only

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[a] It is true that several authors have recently contested the legitimacy of extending the absorption law of Bragg and Peirce to X-rays.

[a] [B.] Rajewsky, *Fortschritte auf dem Gebiet der Roentgenstrahlung*, **35** (1926), 262.

is the number of collision electrons thereby multiplied, but, further, the primary quantum, reduced by successive collisions takes on values for which the absorption with emission of photoelectrons becomes more and more probable.

These facts have an important repercussion on the technique of X-ray therapy. Certain authors had, in fact, denied the usefulness of producing very high-voltage apparatus providing X-rays of very high frequency and very high penetrating power, whose use is otherwise convenient owing to the uniformity of irradiation they allow one to attain. If these rays had been devoid of efficacy, one would have had to give up on their use. Such is not the case if one adopts the point of view of the Compton effect, and it is then legitimate to direct the technique towards the use of high voltages.

Another interesting point of view to examine is that of the emission of β-rays by radioactive bodies. Professor Compton has pointed out that among the β-rays of secondary origin, some could be collision electrons produced by the scattering of the primary γ-rays on the electrons contained in the matter they traverse.

It is in an effect of this type that Thibaud thinks one may find the explanation for the appearance of the magnetic spectra of the secondary γ-rays. These spectra are composed of lines that may be attributed to groups of photoelectrons of the same speed, each of which is emitted by absorption in a thin metallic envelope of a group of homogeneous γ-rays emitted by a radioelement contained in this envelope. Each line of photoelectric origin is accompanied by a band beginning at the line itself and extending towards the region of low velocities. Thibaud thinks that this band could be due to photoelectrons expelled from the screen by those γ-rays that, in this same screen, had suffered the Compton effect with reduction of frequency. This interpretation appears plausible; however, in order to prove it, it would be necessary to study the structure of the band and find in the same spectrum the band that may be attributed to the collision electrons corresponding to the scattered γ-rays.

An analogous problem arises regarding the emission of β-rays by radioactive bodies with negligible thickness, so as to eliminate, as far as possible, the secondary effects due to the supports and envelopes. One then observes a magnetic spectrum that may be attributed to the radioelement alone and consisting either of a continuous band, or of the superposition of a continuous spectrum and a line spectrum. The latter has received a satisfactory interpretation in some recent papers (L. Meitner, Ellis, Thibaud, etc.).

A line is due to a group of photoelectrons with the same speed expelled from the levels of the radioactive atoms by a group of homogeneous γ-rays produced in their nuclei. This effect is called 'internal conversion', since one assumes that the quantum emitted by an atomic nucleus is reabsorbed in the electron cloud [enveloppe électronique] of the same atom. The great majority of observed lines find their explanation in this hypothesis.

The interpretation of the continuous spectrum appears to present more difficulties. Some authors attribute it only to the primary β-rays, while others consider the possibility of a secondary origin and invoke the Compton effect as a possible cause of its production (L. Meitner). This would be an 'internal' Compton effect, such that a γ-ray emitted from the nucleus of an atom would experience a collision with one of the weakly bound electrons at the periphery of the same atom. If that were the case, the velocity distribution of the emitted collision electrons would not be arbitrary, but would have to conform to the predictions of Compton's theory.

I have closely examined this problem, which has a very complex appearance.[a] Each group of homogeneous γ-rays is accompanied by scattered γ-rays, so that in the diffraction spectrum of the γ-rays, each line should experience a broadening of 0.0485 Å units. The experiments on the diffraction of γ-rays are difficult and not very numerous; so far the broadening effect has not been reported.

Each homogeneous group of γ-rays must correspond to a group of collision electrons, whose velocity varies continuously from zero to an upper limit derived from Compton's theory and which in the magnetic spectrum corresponds to a band bounded sharply on the side of the large velocities. The same group of γ-rays may correspond to further groups of photoelectrons expelled from the different levels K, L, etc. of the atom through internal absorption of the scattered γ-rays. For each group of photoelectrons, the velocity of emission lies between two well-defined limits. The upper limit corresponds to the surplus energy of the primary γ-rays with respect to the extraction work W characteristic of the given level; the lower limit corresponds to the surplus energy, with respect to the same work, of the γ-rays scattered in the direction opposite to that of the primary rays, and having experienced because of that the highest loss of frequency. In the magnetic spectrum, each group of photoelectrons will be represented by a band equally well bounded on the side of the

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[a] [M.] Curie, *Le Journal de Physique et le Radium*, **7** (1926), 97.

large and of the small velocities, with the same difference between the extreme energies for each band.

It is easy to see that in the same magnetic spectrum the different bands corresponding to the same group of γ-rays may partially overlap, making it difficult to analyse the spectrum comparing the distribution of β-rays with that predicted by theory. For substances emitting several groups of γ-rays, the difficulty must become considerable, unless there are large differences in their relative effectiveness in producing the desired effect. Let us also point out that the continuous spectrum due to the Compton effect may be superposed with a continuous spectrum independent of this effect (that may be attributed for instance to the primary β-rays).

Examination of the experimental data available so far does not yet allow one to draw conclusions convincingly. Most of the spectra are very complex, and their precise study with respect to the energy distributions of the β-rays will require very detailed work. In certain simple spectra such as that of the β-rays of RaD, one observes lines of photoelectric origin that may be attributed to a single group of monochromatic γ-rays. These lines form the upper edge of bands extending towards low velocities and probably arising from photoelectrons produced by the scattered γ-rays. In certain magnetic spectra obtained from the β-rays of mesothorium 2 in the region of low velocities, one notices in the continuous spectrum a gap that might correspond, for the group of primary γ-rays with 58 kilovolts, to the separation between the band due to the collision electrons and that due to the photoelectrons of the scattered γ-rays.[a]

MR SCHRÖDINGER, at the invitation of Mr Ehrenfest, draws on the blackboard in coloured chalk the system of four wave trains by which he has tried to represent the Compton effect in an anschaulich way [d'une façon intuitive][b] (*Ann. d. Phys.* 4th series, vol. **82** (1927), 257).[c]

MR BOHR. — The simultaneous consideration of two systems of waves has not the aim of giving a causal theory in the classical sense, but one can show that it leads to a symbolic analogy. This has been studied in particular by Klein. Furthermore, it has been possible to treat the

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[a] D. K. Yovanovitch and A. Prola, *Comptes Rendus*, **183** (1926), 878.

[b] For discussions of the notion of Anschaulichkeit, see sections 3.4.7, 4.6 and 8.3 (*eds.*).

[c] Schrödinger (1928, p. x) later remarked on a mistake pointed out to him by Ehrenfest in the figure as published in the original paper (*eds.*).

problem in more depth through the way Dirac has formulated Schrödinger's theory. We find here an even more advanced renunciation of Anschaulichkeit [intuitivité], a fact very characteristic of the symbolic methods in quantum theory.

Mr Lorentz. — Mr Schrödinger has shown how one can explain the Compton effect in wave mechanics. In this explanation one considers the waves associated with the electron (e) and the photon (ph), before (1) and after (2) the encounter. It is natural to think that, of these four systems of waves e_1, ph_1, e_2 and ph_2, the latter two are produced by the encounter. But they are not *determined* by e_1 and ph_1, because one can for example choose arbitrarily the direction of e_2. Thus, for the problem to be well-defined, it is not sufficient to know e_1 and ph_1; another piece of data is necessary, just as in the case of the collision of two elastic balls one must know not only their initial velocities but also a parameter that determines the greater or lesser eccentricity of the collision, for instance the angle between the relative velocity and the common normal at the moment of the encounter. Perhaps one could introduce into the explanation given by Mr Schrödinger something that would play the role of this accessory parameter.

Mr Born. — I think it is easy to understand why three of the four waves have to be given in order for the process to be determined; it suffices to consider analogous circumstances in the classical theory. If the motions of the two particles approaching each other are given, the effect of the collision is not yet determined; it can be made determinate by giving the position of closest approach or an equivalent piece of data. But in wave mechanics such microscopic data are not available. That is why it is necessary to prescribe the motion of one of the particles after the collision, if one wants the motion of the second particle after the collision to be determined. But there is nothing surprising in this, everything being exactly as in classical mechanics. The only difference is that in the old theory one introduces microscopic quantities, such as the radii of the atoms that collide, which are eliminated from subsequent calculations, while in the new theory one avoids the introduction of these quantities.

Notes to the translation

1 The words ‘réels ou apparents’ are present only in the French version.
2 The English version has only four headings (starting with ‘(1) How are the waves produced?’), and accordingly omits the next section, on ‘The problem of the ether’, and later references to the ether.
3 [d’après les résultats de l’étude des spectres]
4 The English edition distinguishes sections and subsections more systematically than the French edition, and in this and other small details of layout we shall mostly follow the former.
5 [rappeler qu’il existe une théorie dans laquelle]
6 The words ‘le professeur’ are present only in the French edition.
7 [‘élément de radiation’ ou ‘quantum de lumière’]
8 This section is present only in the French version.
9 [énergie ondulatoire]
10 The original footnote gives page ‘145’.
11 The English edition has ‘**213**’.
12 [à part une constante]
13 [perfectionnée]
14 The second part of the footnote is printed only in the English edition.
15 The French edition here includes the clause ‘qui a été faite’.
16 Here and in several places in the following, the French edition has ‘simple’ where the English one has ‘single’.
17 [l’action directe du vecteur électrique de l’onde, prise dans le sens ordinaire]
18 [dans la direction de propagation de l’onde]
19 [puisque]
20 This footnote is only present in the English edition.
21 The preceding clause is only present in the French edition.
22 [la période de l’électron dans son mouvement orbital]
23 In the English edition this reads: ‘The fact that the photoelectrons receive the momentum of the incident photon’.
24 [sur les rayons X secondaires et les rayons γ]
25 This footnote appears only in the French edition.
26 [ne fournirent aucune preuve de l’existence d’un spectre de raies]
27 This word is missing in the French edition.
28 The English edition reads ‘$(1 + x)$’.
29 [prédit]
30 [on eut quelque confiance dans les électrons de recul]
31 The French edition uses ρ instead of τ in the text, but uses τ in the discussion (where ρ is used for matter density). The English edition uses t.
32 Footnote mark missing in the French edition.
33 The French edition gives (4).
34 The French edition gives (2).
35 This footnote is present only in the English edition.
36 This word is missing in the French edition.
37 The French edition reads ‘J. Waller’.
38 The page numbers for Wentzel appear only in the French edition.
39 The English edition describes only three experiments, omitting the

section on the composite photoelectric effect as well as references to it later.
40 This section is present only in the French edition.
41 This and the next section are of course numbered (2) and (3) in the English edition.
42 [rayonnement de fluorescence]
43 [une expérience cruciale entre les deux points de vue a été imaginée et brillamment réalisée]
44 The French edition includes also 'en même temps'.
45 [d'une façon appropriée]
46 The French edition has 'θ'.
47 The words 'photons, c'est-à-dire' are present only in the French edition.
48 [au phénomène de la diffusion par les électrons individuels]
49 [ou]
50 [avec certitude]
51 [physiologique]
52 This sentence is only printed in the French edition.
53 The English edition omits reference to the ether and continues directly with 'affords no adequate picture'.
54 The English edition continues directly with: 'seems powerless'.
55 The English edition continues: 'is also contrary to'.
56 In the English edition this reads simply: 'According to the photon theory, the production .... '.
57 In the English edition: 'three'.
58 [processus]
59 Overbars have been added.
60 The French text reads 'avec moins d'intensité sur la charge ayant la plus petite masse'. This is evidently an error: the (transverse) velocity of the charges stems from the electric field, which imparts the larger velocity to the charge with the smaller mass, which therefore experiences the larger magnetic force.
61 The printed text reads '$\varphi_\alpha \omega$'.
62 Brackets in the exponent added.

The new dynamics of quanta[a]

By Mr Louis de BROGLIE

I. — Principal points of view[b]

1. *First works of Mr Louis de Broglie* [1]. — In his first works on the Dynamics of Quanta, the author of the present report started with the following idea: taking the existence of elementary corpuscles of matter and radiation as an experimental fact, these corpuscles are supposed to be endowed with a periodicity. In this way of seeing things, one no longer conceives of the 'material point' as a static entity pertaining to only a tiny region of space, but as the centre of a periodic phenomenon spread all around it.

Let us consider, then, a completely isolated material point and, in a system of reference attached to this point, let us attribute to the postulated periodic phenomenon the appearance of a stationary wave defined by the function

$$u(x_0,\ y_0,\ z_0,\ t_0) = f(x_0,\ y_0,\ z_0)\cos 2\pi\nu_0 t_0 \ .$$

In another Galilean system x, y, z, t, the material point will have a rectilinear and uniform motion with velocity $v = \beta c$. Simple application

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[a] Our translation of the title ('La nouvelle dynamique des quanta') reflects de Broglie's frequent use of the word 'quantum' to refer to a (pointlike) particle, an association that would be lost if the title were translated as, for example, 'The new quantum dynamics' (*eds.*).

[b] On beginning this exposition, it seems right to underline that Mr Marcel Brillouin was the true precursor of wave Mechanics, as one may realise by referring to the following works: *C. R.* **168** (1919), 1318; **169** (1919), 48; **171** (1920), 1000. — *Journ. Physique* **3** (1922), 65.

of the Lorentz transformation shows that, as far as the phase is concerned, in the new system the periodic phenomenon has the appearance of a plane wave propagating in the direction of motion whose frequency and phase velocity are

$$\nu = \frac{\nu_0}{\sqrt{1-\beta^2}} \ , \quad V = \frac{c^2}{v} = \frac{c}{\beta} \ .$$

The appearance of this phase propagation with a speed superior to c, as an immediate consequence of the theory of Relativity, is quite striking.

There exists a noteworthy relation between v and V. The formulas giving ν and V allow us in fact to define a refractive index of the vacuum, for the waves of the material point of proper frequency ν_0, by the dispersion law

$$n = \frac{c}{V} = \sqrt{1 - \frac{\nu_0^2}{\nu^2}} \ .$$

One then easily shows that

$$\frac{1}{v} = \frac{1}{c}\frac{\partial(n\nu)}{\partial\nu} \ ,$$

that is, that the velocity v of the material point is equal to the group velocity corresponding to the dispersion law.

With the free material point being thus defined by wave quantities, the dynamical quantities must be related back to these. Now, since the frequency ν transforms like an energy, the obvious thing to do is to assume the quantum relation

$$W = h\nu \ ,$$

a relation that is valid in all systems, and from which one derives the undulatory definition of the proper mass m_0

$$m_0c^2 = h\nu_0 \ .$$

Let us write the function representing the wave in the system x, y, z, t in the form

$$u(x,y,z,t) = f(x,y,z,t)\cos\frac{2\pi}{h}\varphi(x,y,z,t) \ .$$

Denoting by W and p the energy and momentum, one easily shows that one has[a,b]

$$W = \frac{\partial \varphi}{\partial t} \ , \quad \overrightarrow{p} = -\overrightarrow{\text{grad}\ \varphi} \ .$$

The function φ is then none other than the Jacobi function.[a] One deduces from this that, in the case of uniform rectilinear motion, the principles of least action and of Fermat are identical.

To look for a generalisation of these results, let us now assume that the material point moving in a field derived from a potential function $F(x, y, z, t)$ is represented by the function

$$u(x, y, z, t) = f(x, y, z, t) \cos \frac{2\pi}{h} \varphi(x, y, z, t) \ ,$$

where φ is the Jacobi function of the old Dynamics. This assimilation of the phase into the Jacobi function then leads us to assume the following two relations, which establish a general link between mechanical quantities and wave quantities:

$$W = h\nu = \frac{\partial \varphi}{\partial t} \ , \quad \overrightarrow{p} = \frac{h\nu}{V} = -\overrightarrow{\text{grad}\ \varphi} \ .$$

One then deduces that, for the waves of the new Mechanics, the space occupied by the field has a refractive index

$$n = \sqrt{\left(1 - \frac{F}{h\nu}\right)^2 - \frac{\nu_0^2}{\nu^2}} \ .$$

Hamilton's equations show in addition that, here again, the velocity of the moving body is equal to the group velocity.[b]

These conceptions lead to an interpretation of the stability conditions introduced by quantum theory. If, indeed, one considers a closed trajectory, the phase must be a single-valued function along this curve, and as a result one is led to write the Planck condition[1]

$$\oint (p \cdot dl) = k \cdot h \quad (k \text{ integer}) \ .$$

The Sommerfeld conditions for quasi-periodic motions may also be derived. The phenomena of quantum stability thus appear to be analogous to phenomena of resonance, and the appearance of whole numbers here

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[a] These are the relativistic guidance equations of de Broglie's early pilot-wave theory of 1923–24, for the special case of a free particle (*eds.*).

[b] The vector '$\overrightarrow{\text{grad}\ \varphi}$' is the vector whose components are $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$.

[a] Usually called the Hamilton-Jacobi function (*eds.*).

[b] In the case of motion of a point charge in a magnetic field, space behaves like an anisotropic medium (see *Thesis*, p. 39).

becomes as natural as in the theory of vibrating strings or plates. Nevertheless, as we shall see, the interpretation that has just been recalled still constitutes only a first approximation.

The application of the new conceptions to corpuscles of light leads to difficulties if one considers their proper mass to be finite. One avoids these difficulties by assuming that the properties of the corpuscles of light are deduced from those of ordinary material points by letting the proper mass tend to zero. The two speeds v and V then both tend to c, and in the limit one obtains the two fundamental relations of the theory of light quanta

$$h\nu = W \ , \quad \frac{h\nu}{c} = p \ ,$$

with the aid of which one can account for Doppler effects, radiation pressure, the photoelectric effect and the Compton effect.

The new wave conception of Mechanics leads to a new statistical Mechanics, which allows us to unify the kinetic theory of gases and the theory of blackbody radiation into a single doctrine. This statistics coincides with that proposed independently by Mr Bose [2]; Mr Einstein [3] has shown its scope and clarified its significance. Since then, numerous papers [4] have developed it in various directions.

Let us add a few remarks. First, the author of this report has always assumed that the material point occupies a well-defined position in space. As a result, the amplitude f should contain a singularity or at the very least have abnormally high values in a very small region. But, in fact, the form of the amplitude plays no role in the results reviewed above. Only the phase intervenes: hence the name *phase waves* originally given to the waves of the new Mechanics.

On the other hand, the author, after having reduced the old forms of Dynamics to geometrical Optics, realised clearly that this was only a first stage. The existence of diffraction phenomena appeared to him to require the construction of a new Mechanics 'which would be to the old Mechanics (including that of Einstein) what wave Optics is to geometrical Optics'.[a] It is Mr Schrödinger who has had the merit of definitively constructing the new doctrine.

2. *The work of Mr E. Schrödinger* [5]. — Mr Schrödinger's fundamental idea seems to have been the following: the new Mechanics must begin from wave equations, these equations being constructed in such a way

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[a] *Revue Générale des Sciences*, 30 November 1924, p. 633.

that in each case the phase of their sinusoidal solutions should be a solution of the Jacobi equation in the approximation of geometrical Optics.

Instead of considering waves whose amplitude contains a singularity, Mr Schrödinger systematically looks at waves of classical type, that is to say, waves whose amplitude is a continuous function. For him, the waves of the new Mechanics are therefore represented by functions Ψ that one can always write in the canonical form

$$\Psi = a \cos \frac{2\pi}{h} \varphi \ ,$$

a being a continuous function and φ being *in the first approximation* a solution of the Jacobi equation. We may understand the words 'in the first approximation' in two different ways: first, if the conditions that legitimate the use of geometrical Optics are realised, the phase φ will obey the equation called the equation of geometrical Optics, and this equation will have to be identical to that of Jacobi; second, one must equally recover the Jacobi equation if one makes Planck's constant tend to zero, because we know in advance that the old Dynamics must then become valid.

Let us first consider the case of the motion of a single material point in a static field derived from the potential function $F(x, y, z)$. In his first Memoir Schrödinger shows that the wave equation, at least in the approximation of Newton's Mechanics, is in this case

$$\triangle \Psi + \frac{8\pi^2 m_0}{h^2}(E - F)\Psi = 0 \ .$$

It is also just this equation that one arrives at beginning from the dispersion law noted in the first section.

Having obtained this equation, Mr Schrödinger used it to study the quantisation of motion at the atomic scale (hydrogen atom, Planck oscillator, etc.). He made the following fundamental observation: in the problems considered in micromechanics, the approximations of geometrical Optics are no longer valid at all. As a result, the interpretation of the quantum conditions proposed by L. de Broglie shows only that the Bohr-Sommerfeld formulas correspond to the approximation of the old Dynamics. To resolve the problem of quantisation rigorously, one must therefore consider the atom as the seat of stationary waves satisfying certain conditions. Schrödinger assumed, as is very natural, that the wave functions must be finite, single-valued and continuous over all space. These conditions define a set of fundamental functions (Eigenfunktionen) for the amplitude, which represent the various stable states of the atomic system being considered. The results obtained have proven that this new quantisation method, to which Messrs Léon Brillouin, G. Wentzel and Kramers [6] have made important contributions, is the correct one.

For Mr Schrödinger, one must look at continuous waves, that is to say, waves whose amplitude does not have any singularities. How can one then represent the 'material point'? Relying on the equality of the velocity of the moving body and the group velocity, Schrödinger sees the material point as a group of waves (Wellenpaket[2]) of closely neighbouring frequencies propagating in directions contained within the interior of a very narrow cone. The material point would then not be really pointlike; it would occupy a region of space that would be at least of the order of magnitude of its wavelength. Since, in intra-atomic phenomena, the domain where motion takes place has dimensions of the order of the wavelengths, there the material point would no longer be defined at all; for Mr Schrödinger, the electron in the atom is in some sense 'smeared out' ['fondu'], and one can no longer speak of its position or velocity. This manner of conceiving of material points seems to us to raise many difficulties; if, for example, the quantum of ultraviolet light occupies a volume whose dimensions are of the order of its wavelength, it is quite difficult to conceive that this quantum could be absorbed by an atom of dimensions a thousand times smaller.

Having established the wave equation for a material point in a static field, Mr Schrödinger then turned to the Dynamics of many-body systems [la Dynamique des systèmes]. Still limiting himself to the Newtonian[a] approximation, and inspired by Hamilton's ideas, he arrived at the following statement: Given an isolated system whose potential energy is $F(q_1, q_2, ..., q_n)$, the kinetic energy is a homogeneous quadratic form in the momenta p_k and one may write

$$2T = \sum_{kl} m^{kl} p_k p_l \ ,$$

the m^{kl} being functions of the q. If m denotes the determinant $\left|m^{kl}\right|$ and if E is the constant of energy in the classical sense, then according to Schrödinger one must begin with the wave equation

$$m^{+\frac{1}{2}} \sum_{kl} \frac{\partial}{\partial q_k} \left[m^{-\frac{1}{2}} m^{kl} \frac{\partial \Psi}{\partial q^l} \right] + \frac{8\pi^2}{h^2}(E - F)\Psi = 0 \ ,$$

which describes the propagation of a wave in the configuration space

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[a] That is, nonrelativistic (*eds.*).

constructed by means of the variables q. Setting

$$\Psi = a \cos \frac{2\pi}{h} \varphi \ ,$$

and letting h tend to zero, in the limit one indeed recovers the Jacobi equation

$$\frac{1}{2} \sum_{kl} m^{kl} \frac{\partial \varphi}{\partial q^k} \frac{\partial \varphi}{\partial q^l} + F = E \ .$$

To quantise an atomic system, one will here again determine the fundamental functions of the corresponding wave equation.

We cannot recall here the successes obtained by this method (papers by Messrs Schrödinger, Fues,[3] Manneback [7], etc.), but we must insist on the difficulties of a conceptual type that it raises. Indeed let us consider, for simplicity, a system of N material points each possessing three degrees of freedom. The configuration space is in an essential way formed by means of the coordinates of the points and yet Mr Schrödinger assumes that in atomic systems material points no longer have a clearly defined position. It seems a little paradoxical to construct a configuration space with the coordinates of points that do not exist. Furthermore, if the propagation of a wave in space has a clear physical meaning, it is not the same as the propagation of a wave in the abstract configuration space, for which the number of dimensions is determined by the number of degrees of freedom of the system. We shall therefore have to return later to the exact meaning of the Schrödinger equation for many-body systems.

By a transformation of admirable ingenuity, Mr Schrödinger has shown that the quantum Mechanics invented by Mr Heisenberg and developed by Messrs Born, Jordan, Pauli, etc., can be translated into the language of wave Mechanics. By comparison with Heisenberg's matrix elements, he was able to derive the expression for the mean charge density of the atom from the functions Ψ, an expression to which we shall return later.

The Schrödinger equations are not relativistic. For the case of a single material point, various authors [8] have given a more general wave equation that is in accord with the principle of Relativity. Let e be the electric charge of the point, $\mathcal{V}$ and $\overrightarrow{A}$ the two electromagnetic potentials.

The equation that the wave Ψ, written in complex form, must satisfy is[a]

$$\Box\Psi + \frac{4\pi i}{h}\frac{e}{c}\left[\frac{\mathcal{V}}{c}\frac{\partial\Psi}{\partial t} + \sum_{xyz} A_x \frac{\partial\Psi}{\partial x}\right] - \frac{4\pi^2}{h^2}\left[m_0^2c^2 - \frac{e^2}{c^2}(\mathcal{V}^2 - A^2)\right]\Psi = 0 \ .$$

As Mr O. Klein [9] and then the author [10] have shown, the theory of the Universe with five dimensions allows one to give the wave equation a more elegant form in which the imaginary terms, whose presence is somewhat shocking for the physicist, have disappeared.

We must also make a special mention of the beautiful Memoirs in which Mr De Donder [11] has connected the formulas of wave Mechanics to his general theory of Einsteinian Gravity.

3. *The ideas of Mr Born* [12]. — Mr Born was struck by the fact that the continuous wave functions Ψ do not allow us to say *where* the particle whose motion one is studying is and, rejecting the concept of the Wellenpaket, he considers the waves Ψ as giving only a statistical representation of the phenomena. Mr Born seems even to abandon the idea of the determinism of individual physical phenomena: the Quantum Dynamics, he wrote in his letter to *Nature*, 'would then be a singular fusion of mechanics and statistics .... . A knowledge of Ψ enables us to follow the course of a physical process in so far as it is quantum mechanically determinate: not in a causal sense, but in a statistical one'.[4]

These conceptions were developed in a mathematical form by their author, in Memoirs of fundamental interest. Here, by way of example, is how he treats the collision of an electron and an atom. He writes the Schrödinger equation for the electron-atom system, and he remarks that before the collision, the wave Ψ must be expressed by the product of the fundamental function[a] representing the initial state of the atom and the plane wave function corresponding to the uniform rectilinear motion of the electron. During the collision, there is an interaction between the electron and the atom, an interaction that appears in the wave equation as the mutual potential energy term. Starting from the initial form of Ψ, Mr Born derives by methods of successive approximation its final form after the collision, in the case of an elastic collision, which does not modify the internal state of the atom, as well as in the case of an inelastic collision, where the atom passes from one stable state to another taking energy from or yielding it to the electron. According to Mr Born, the

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[a] This is the complex, time-dependent Klein-Gordon equation in an external electromagnetic field (*eds.*).

[a] That is, eigenfunction (*eds.*).

final form of Ψ determines the probability that the collision may produce this or that result.

The ideas of Mr Born seem to us to contain a great deal of truth, and the considerations that shall now be developed show a great analogy with them.

II. — PROBABLE MEANING OF THE CONTINUOUS WAVES Ψ [13]

4. *Case of a single material point in a static field.* — The body of experimental discoveries made over forty years seems to require the idea that matter and radiation possess an atomic structure. Nevertheless, classical optics has with immense success described the propagation of light by means of the concept of continuous waves and, since the work of Mr Schrödinger, also in wave Mechanics one always considers continuous waves which, not showing any singularities, do not allow us to define the material point. If one does not wish to adopt the hypothesis of the 'Wellenpaket', whose development seems to raise difficulties, how can one reconcile the existence of pointlike elements of energy with the success of theories that consider the waves Ψ? What link must one establish between the corpuscles and the waves? These are the chief questions that arise in the present state of wave Mechanics.

To try to answer this, let us begin by considering the case of a single corpuscle carrying a charge e and moving in an electromagnetic field[a] defined by the potentials $\mathcal{V}$ and $\overrightarrow{A}$. Let us suppose first that the motion is one for which the old Mechanics (in relativistic form) is sufficient. If we write the wave Ψ in the canonical form

$$\Psi = a \cos \frac{2\pi}{h} \varphi \ ,$$

the function φ is then, as we have seen, the Jacobi function, and the velocity of the corpuscle is defined by the formula of Einsteinian Dynamics

$$\overrightarrow{v} = -c^2 \frac{\overrightarrow{\text{grad}\ \varphi} + \frac{e}{c}\overrightarrow{A}}{\frac{\partial \varphi}{\partial t} - e\mathcal{V}} \ . \tag{I}$$

We propose to assume by induction that this formula is still valid when the old Mechanics is no longer sufficient, that is to say when φ is no longer a solution of the Jacobi equation.[b] If one accepts this hypothesis, which appears justified by its^5 consequences, the formula (I)

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[a] Here we leave aside the case where there also exists a gravitational field. Besides, the considerations that follow extend without difficulty to that case.

[b] Mr De Donder assumes equation (I) as we do, but denoting by φ not the phase of

completely determines the motion of the corpuscle *as soon as one is given its position at an initial instant.* In other words, the function φ, just like the Jacobi function of which it is the generalisation, determines a whole class of motions, and to know which of these motions is actually described it suffices to know the initial position.

Let us now consider a whole cloud of corpuscles, identical and without interaction, whose motions, determined by (I), correspond to the same function φ but differ in the initial positions. Simple reasoning shows that if the density of the cloud at the initial moment is equal to

$$Ka^2\left(\frac{\partial\varphi}{\partial t}-e\mathcal{V}\right)\ ,$$

where K is a constant, it will subsequently remain constantly given by this expression. We can state this result in another form. Let us suppose there be only a single corpuscle whose initial position we ignore; from the preceding, the *probability for its presence* [sa *probabilité de présence*] at a given instant in a volume $d\tau$ of space will be

$$\pi\, d\tau = Ka^2\left(\frac{\partial\varphi}{\partial t}-e\mathcal{V}\right) d\tau\ . \tag{II}$$

In brief, in our hypotheses, each wave Ψ determines a 'class of motions', and each one of these motions is governed by equation (I) when one knows the initial position of the corpuscle. If one ignores this initial position, the formula (II) gives the probability for the presence of the corpuscle in the element of volume $d\tau$ at the instant t. The wave Ψ then appears as both a *pilot wave* (Führungsfeld of Mr Born) and a *probability wave.* Since the motion of the corpuscle seems to us to be strictly determined by equation (I), it does not seem to us that there is any reason to renounce believing in the determinism of individual physical phenomena,[a] and it is in this that our conceptions, which are very similar in other respects to those of Mr Born, appear nevertheless to differ from them markedly.

Let us remark that, if one limits oneself to the Newtonian approximation, in (I) and (II) one can replace: $\frac{\partial\varphi}{\partial t}-e\mathcal{V}$ by m_0c^2, and one obtains the simplified forms

$$\overrightarrow{v} = -\frac{1}{m_0}\left(\overrightarrow{\mathrm{grad}\ \varphi}+\frac{e}{c}\overrightarrow{A}\right)\ , \tag{I$'$}$$

the wave, but the classical Jacobi function. As a result his theory and ours diverge as soon as one leaves the domain where the old relativistic Mechanics is sufficient.

[a] Here, that is, of the motion of individual corpuscles.

$$\pi = \text{const} \cdot a^2 \ . \tag{II$'$}$$

There is one case where the application of the preceding ideas is done in a remarkably clear form: when the initial motion of the corpuscles is uniform and rectilinear in a region free of all fields. In this region, the cloud of corpuscles we have just imagined may be represented by the homogeneous plane wave[6]

$$\Psi = a \cos \frac{2\pi}{h} W \left(t - \frac{vx}{c^2} \right) \ ;$$

here a is a constant, and this means that a corpuscle has the same probability to be at any point of the cloud. The question of knowing how this homogeneous plane wave will behave when penetrating a region where a field is present is analogous to that of determining the form of an initially plane light wave that penetrates a refracting medium. In his Memoir 'Quantenmechanik der Stossvorgänge', Mr Born has given a general method of successive approximation to solve this problem, and Mr Wentzel [14] has shown that one can thus recover the Rutherford formula for the deflection of β-rays by a charged centre.

We shall present yet another observation on the Dynamics of the material point such as results from equation (I): for the material point one can always write the equations of the Dynamics of Relativity even when the approximation of the old mechanics is not valid, on condition that one attributes to the body a variable proper mass M_0 given by the formula

$$M_0 = \sqrt{m_0^2 - \frac{h^2}{4\pi^2 c^2} \frac{\Box a}{a}} \ .$$

5. *The interpretation of interference.* — The new Dynamics allows us to interpret the phenomena of wave Optics in exactly the way that was foreseen, a long time ago now, by Mr Einstein.[a] In the case of light, the wave Ψ is indeed the light wave of the classical theories.[b,c] If we consider the propagation of light in a region strewn with fixed obstacles, the propagation of the wave Ψ will depend on the nature and arrangement of these obstacles, but the frequency $\frac{1}{h}\frac{\partial \varphi}{\partial t}$ will not vary (no Doppler

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[a] Cf. chapter 9 (*eds.*).

[b] We then consider Ψ as the 'light variable' without at all specifying the physical meaning of this quantity.

[c] By 'classical theories' de Broglie seems to mean scalar wave optics. In the general discussion (p. 508), de Broglie states that the physical nature of Ψ for photons is unknown (*eds.*).

effect). The formulas (I) and (II) will then take the form

$$\vec{v} = -\frac{c^2}{h\nu}\overrightarrow{\mathrm{grad}\ \varphi}\ ; \qquad \pi = \mathrm{const}{\cdot}a^2\ .$$

The second of these formulas shows immediately that the bright and dark fringes predicted by the new theory will coincide with those predicted by the old. To record the fringes, for example by photography, one can do an experiment of short duration with intense irradiation, or an experiment of long duration with feeble irradiation (Taylor's experiment); in the first case one takes a mean in space, in the second case a mean in time, but if the light quanta do not act on each other the statistical result must evidently be the same.

Mr Bothe [15] believed he could deduce, from certain experiments on the Compton effect in a field of interference, the inexactitude of the first formula written above, the one giving the velocity of the quantum, but in our opinion this conclusion can be contested.

6. *The energy-momentum tensor of the waves* Ψ. — In one of his Memoirs [16], Mr Schrödinger gave the expression for the energy-momentum tensor in the interior of a wave Ψ.[a] Following the ideas expounded here, the wave Ψ represents the motion of a cloud of corpuscles; examining the expression given by Schrödinger and taking into account the relations (I) and (II), one then perceives that it decomposes into one part giving the energy and momentum of the particles, and another that can be interpreted as representing a state of stress existing in the wave around the particles. These stresses are zero in the states of motion consistent with the old Dynamics; they characterise the new states predicted by wave Mechanics, which thus appear as 'constrained states' of the material point and are intimately related to the variability of the proper mass M_0. Mr De Donder has also drawn attention to this fact, and he was led to denote the amplitude of the waves that he considered by the name of 'internal stress potential'.

The existence of these stresses allows one to explain how a mirror reflecting a beam of light suffers a radiation pressure, even though according to equation (I), because of interference, the corpuscles of light do not 'strike' its surface.[b]

7. *The dynamics of many-body systems.* — We must now examine how

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[a] Cf. Schrödinger's report, section II (*eds.*).

[b] Cf. Brillouin's example in the discussion at the end of de Broglie's lecture (*eds.*).

these conceptions may serve to interpret the wave equation proposed by Schrödinger for the Dynamics of many-body systems. We have pointed out above the two difficulties that this equation raises. The first, relating to the meaning of the variables that serve to construct the configuration space, disappears if one assumes that the material points always have a quite definite position. The second difficulty remains. It appears to us certain that if one wants to *physically* represent the evolution of a system of N corpuscles, one must consider the propagation of N waves in space, each of the N propagations being determined by the action of the $N-1$ corpuscles connected to the other waves.[a] Nevertheless, if one focusses one's attention only on the corpuscles, one can represent their states by a point in configuration space, and one can try to relate the motion of this representative point to the propagation of a fictitious wave Ψ in configuration space. It appears to us very probable that the wave[b]

$$\Psi = a(q_1,\ q_2,\ ...,\ q_n)\cos\frac{2\pi}{h}\varphi(t,\ q_1,\ ...,\ q_n)\ ,$$

a solution of the Schrödinger equation, is only a fictitious wave which, in the *Newtonian approximation*, plays for the representative point of the system in configuration space the same role of pilot wave and of probability wave that the wave Ψ plays in ordinary space in the case of a single material point.

Let us suppose the system to be formed of N points having for rectangular coordinates

$$x_1^1,\ x_2^1,\ x_3^1,\ ...,\ x_1^N,\ x_2^N,\ x_3^N\ .$$

In the configuration space formed by means of these coordinates, the representative point of the system has for [velocity] components along the axis x_i^k

$$v_{x_i^k} = -\frac{1}{m_k}\frac{\partial\varphi}{\partial x_i^k}\ ,$$

m_k being the mass of the kth corpuscle. This is the relation that replaces (I′) for many-body systems. From this, one deduces that the probability for the presence of the representative point in the element of volume $d\tau$

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[a] Cf. section 2.4 (*eds.*).

[b] The amplitude a is time-independent because de Broglie is assuming the time-independent Schrödinger equation. Later in his report, de Broglie applies his dynamics to a non-stationary wave function as well, for the case of an atomic transition (*eds.*).

of configuration space is

$$\pi \ d\tau = \text{const}\cdot a^2 \ d\tau \ .$$

This new relation replaces relation (II′) for many-body systems. It fully accords, it seems to us, with the results obtained by Mr Born for the collision of an electron and an atom, and by Mr Fermi [17] for the collision of an electron and a rotator.[a]

Contrary to what happens for a single material point, it does not appear easy to find a wave Ψ that would define the motion of the system taking Relativity into account.

8. *The waves Ψ in micromechanics.* — Many authors think it is illusory to wonder what the position or the velocity of an electron in the atom is at a given instant. We are, on the contrary, inclined to believe that it is possible to attribute to the corpuscles a position and a velocity even in atomic systems, in a way that gives a precise meaning to the variables of configuration space.

This leads to conclusions that deserve to be emphasised. Let us consider a hydrogen atom in one of its stable states. According to Schrödinger, in spherical coordinates[b] the corresponding function Ψ_n is of the form

$$\Psi_n = F(r,\theta)[A\cos m\alpha + B\sin m\alpha] \ \begin{matrix}\sin\\ \cos\end{matrix} \ \frac{2\pi}{h}W_n t \qquad (m \text{ integer})$$

with

$$W_n = m_0c^2 - \frac{2\pi^2 m_0 e^4}{n^2h^2} \ .$$

If we then apply our formula (I′), we conclude that the electron is motionless in the atom, a conclusion which would evidently be inadmissible in the old Mechanics. However, the examination of various questions and notably of the Zeeman effect has led us to believe that, in its stable states, the H atom must rather be represented by the function

$$\Psi_n = F(r,\theta)\cos\frac{2\pi}{h}\left(W_n t - \frac{mh}{2\pi}\alpha\right) \ ,$$

which, being a linear combination of expressions of the type written

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[a] Cf. the remarks by Born and Brillouin in the discussion at the end of de Broglie's lecture, and the de Broglie-Pauli encounter in the general discussion at the end of the conference (pp. 509 ff.) (*eds.*).

[b] r, radius vector; θ, latitude; α, longitude.

above, is equally acceptable.[a] If this is true the electron will have, from (I′), a uniform circular motion of speed

$$v = \frac{1}{m_0 r}\frac{mh}{2\pi} .$$

It will then be motionless only in states where $m = 0$.

Generally speaking, the states of the atom at a given instant can always be represented by a function

$$\Psi = \sum_n c_n \Psi_n ,$$

the Ψ_n being Schrödinger's Eigenfunktionen. In particular, the state of transition $i \rightarrow j$ during which the atom emits the frequency ν_{ij} would be given by (this appears to be in keeping with Schrödinger's ideas)

$$\Psi = c_i \Psi_i + c_j \Psi_j ,$$

c_i and c_j being two functions of time that change very slowly compared with the trigonometric factors of the Ψ_n, the first varying from 1 to 0 and the second from 0 to 1 during the transition. Writing the function Ψ in the canonical form $a \cos \frac{2\pi}{h}\varphi$, which is always possible, formula (I′) will give the velocity of the electron during the transition, if one assumes the initial position to be given. So it does not seem to be impossible to arrive in this way at a visual representation of the transition.[a]

Let us now consider an ensemble of hydrogen atoms that are all in the same state represented by the same function

$$\Psi = \sum_n c_n \Psi_n .$$

The position of the electron in each atom is unknown to us, but if, in our imagination, we superpose all these atoms, we obtain a *mean atom* where the probability for the presence of one of the electrons in an element of volume $d\tau$ will be given by the formula (II),[b] K being determined by the fact that the total probability for all the possible positions must be

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[a] In his memoir, 'Les moments de rotation et le magnétisme dans la mécanique ondulatoire' (*Journal de Physique* **8** (1927), 74), Mr Léon Brillouin has implicitly assumed the hypothesis that we formulate in the text.

[a] In this example, de Broglie is applying his dynamics to a case where the wave function Ψ has a time-dependent amplitude a (*eds.*).

[b] In this section on atomic physics ('micromechanics') de Broglie considers the non-relativistic approximation, using the limiting formula (I′) — except in this paragraph where he reverts to the relativistic formulas (I) and (II), for the purpose of comparison with the relativistic formulas for charge and current density obtained by other authors (*eds.*).

equal to unity. The charge density ρ and the current density $\overrightarrow{J} = \rho\overrightarrow{v}$ in the mean atom are then, from (I) and (II),

$$\rho = Kea^2\left(\frac{\partial\varphi}{\partial t} - e\mathcal{V}\right) \ ,$$
$$\overrightarrow{J} = -Kec^2a^2\left(\overrightarrow{\text{grad}\ \varphi} + \frac{e}{c}\overrightarrow{A}\right)$$

and these formulas coincide, apart from notation, with those of Messrs Gordon, Schrödinger and O. Klein [18].

Limiting ourselves to the Newtonian approximation, and for a moment denoting by Ψ the wave written in *complex* form, and by $\bar{\Psi}$ the conjugate function, it follows that

$$\rho = \text{const}{\cdot}a^2 = \text{const}{\cdot}\Psi\bar{\Psi} \ .$$

This is the formula to which Mr Schrödinger was led in reformulating the matrix theory; it shows that the electric dipole moment of the mean atom during the transition $i \to j$ contains a term of frequency ν_{ij}, and thus allows us to interpret Bohr's frequency relation.

Today it appears certain that one can predict the mean energy radiated by an atom by using the Maxwell-Lorentz equations, on condition that one introduces in these equations the mean quantities ρ and $\rho\overrightarrow{v}$ which have just been defined.[a] One can thus give the correspondence principle an entirely precise meaning, as Mr Debye [19] has in fact shown in the particular case of motion with one degree of freedom. It seems indeed that classical electromagnetism can from now on retain only a statistical value; this is an important fact, whose meaning one will have to try to explore more deeply.

To study the interaction of radiation with an ensemble of atoms, it is rather natural to consider a 'mean atom', immersed in a 'mean light' which one defines by a homogeneous plane wave of the vector potential. The density ρ of the mean atom is perturbed by the action of the light and one deduces from this the scattered radiation. This method, which gives good mean predictions, is related more or less directly to the theories of scattering by Messrs Schrödinger and Klein [20], to the theory of the Compton effect by Messrs Gordon and Schrödinger [21], and to the Memoirs of Mr Wentzel [22] on the photoelectric effect and the Compton effect, etc. The scope of this report does not permit us to dwell any further on this interesting work.

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[a] Cf. Schrödinger's report, p. 454, and the ensuing discussion, and section 4.5 (*eds.*).

9. *Conclusions and remarks.* — So far we have considered the corpuscles as 'exterior' to the wave Ψ, their motion being only determined by the propagation of the wave. This is, no doubt, only a provisional point of view: a true theory of the atomic structure of matter and radiation should, it seems to us, *incorporate* the corpuscles in the wave phenomenon by considering singular solutions of the wave equations. One should then show that there exists a correspondence between the singular waves and the waves Ψ, such that the motion of the singularities is connected to the propagation of the waves Ψ by the relation (I).[a] In the case of no [external] field, this correspondence is easily established, but it is not so in the general case.

We have seen that the quantities ρ and $\rho\overrightarrow{v}$ appearing in the Maxwell-Lorentz equations must be calculated in terms of the functions Ψ, but that does not suffice to establish a deep link between the electromagnetic quantities and those of wave Mechanics. To establish this link,[a] one should probably begin with singular waves, for Mr Schrödinger has very rightly remarked that the potentials appearing in the wave equations are those that result from the discontinuous structure of electricity and not those that could be deduced from the functions Ψ.

Finally, we point out that Messrs Uhlenbeck and Goudsmit's hypothesis of the magnetic electron, so necessary to explain a great number of phenomena, has not yet found its place in the scope of wave Mechanics.

III. — EXPERIMENTS SHOWING PRELIMINARY DIRECT EVIDENCE FOR THE NEW DYNAMICS OF THE ELECTRON

10. *Phenomena whose existence is suggested by the new conceptions.* — The ideas that have just been presented lead one to consider the motion of an electron as guided by the propagation of a certain wave. In many usual cases, the old Mechanics remains entirely adequate as a first approximation; but our new point of view, as Elsasser[7] [23] pointed out already in 1925, necessarily raises the following question: 'Could one not observe electron motions that the old Mechanics would be incapable of predicting, and which would therefore be characteristic of wave Mechanics? In other words, for electrons, could one not find the analogue of the phenomena of diffraction and interference?'[b]

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[a] Cf. section 2.3.1 (*eds.*).

[a] The few attempts made till now in this direction, notably by Mr Bateman (*Nature* **118** (1926), 839) and by the author (*Ondes et mouvements*, Chap. VIII, and *C. R.* **184** (1927), 81) can hardly be regarded as satisfactory.

[b] This is not a quotation: these words do not appear in the cited 1925 paper by

These new phenomena, if they exist, must depend on the wavelength of the wave associated with the electron motion. For an electron of speed v, the fundamental formula

$$p = \frac{h\nu}{V}$$

gives

$$\lambda = \frac{V}{\nu} = \frac{h}{p} = \frac{h\sqrt{1-\beta^2}}{m_0 v} \ .$$

If β is not too close to 1, it suffices to write

$$\lambda = \frac{h}{m_0 v} \ .$$

Let $\mathcal{V}$ be the potential difference, expressed in volts, that is capable of imparting the speed v to the electron; numerically, for the wavelength in centimetres, one will have[a]

$$\lambda = \frac{7.28}{v} = \frac{12.25}{\sqrt{\mathcal{V}}} \times 10^{-8} \ .$$

To do precise experiments, it is necessary to use electrons of at least a few volts: from which one has an upper limit for λ of a few angstroms. One then sees that, even for slow electrons, the phenomena being sought are analogous to those shown by X-rays and not to those of ordinary light. As a result, it will be difficult to observe the diffraction of a beam of electrons by a small opening, and if one wishes to have some chance of obtaining diffraction by a grating, one must either consider those natural three-dimensional gratings, the crystals, or use ordinary gratings under a very grazing incidence, as has been done recently for X-rays. On making slow electrons pass through a crystalline powder or an amorphous substance, one could also hope to notice the appearance of rings analogous to those that have been obtained and interpreted in the X-ray domain by Messrs Hull, Debye and Scherrer, Debierne, Keesom and De Smedt, etc.

The exact theoretical prediction of the phenomena to be observed

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Elsasser. Further, it was de Broglie who first suggested electron diffraction, in a paper of 1923 (see section 2.2.1) (*eds.*).

[a] Here we have adopted the following values:

$$h = 6.55 \times 10^{-27} \text{ erg-seconds} \ ,$$
$$m_0 = 9 \times 10^{-28} \text{ gr} \ ,$$
$$e = 4.77 \times 10^{-10} \text{ e.s.u.} \ .$$

along these lines is still not very advanced. Let us consider the diffraction of a beam of electrons with the same velocity by a crystal; the wave Ψ will propagate following the general equation, in which one has to insert the potentials created by the atoms of the crystal considered as centres of force. One does not know the exact expression for these potentials but, because of the regular distribution of atoms in the crystal, one easily realises that the scattered amplitude will show maxima in the directions predicted by Mr von Laue's theory. Because of the role of pilot wave played by the wave Ψ, one must then observe a selective scattering of the electrons in these directions.

Using his methods, Mr Born has studied another problem: that of the collision of a narrow beam of electrons with an atom. According to him, the curve giving the number of electrons that have suffered an inelastic collision as a function of the scattering angle must show maxima and minima; in other words, these electrons will display rings on a screen placed normally to the continuation of the incident beam.

It would still be premature to speak of agreement between theory and experiment; nevertheless, we shall present experiments that have revealed phenomena showing at least broadly the predicted character.

11. *Experiments by Mr E.G. Dymond* [24]. — Without feeling obliged to follow the chronological order, we shall first present Mr Dymond's experiments:

A flask of purified helium contained an 'electron gun', which consisted of a brass tube containing an incandescent filament of tungsten and in whose end a slit was cut. This gun discharged a well-collimated beam of electrons into the gas, with a speed determined by the potential difference (50 to 400 volts) established between the filament and the wall of the tube. The wall of the flask had a slit through which the electrons could enter a chamber where the pressure was kept low by pumping and where, by curving their trajectories, a magnetic field brought them onto a Faraday cylinder.

Mr Dymond first kept the orientation of the gun fixed and measured the speed of the electrons thus scattered by a given angle. He noticed that most of the scattered electrons have the same energy as the primary electrons; they have therefore suffered an elastic collision. Quite a large number of electrons have a lower speed corresponding to an energy loss from about 20 to 55 volts: this shows that they made the He atom pass from the normal state 1^1S to the excited state 2^1S. One also observes a lower proportion of other values for the energy of the scattered electrons;

we shall not discuss the interpretation that Mr Dymond has given them, because what interests us most here is the variation of the number of scattered particles with the scattering angle θ. To determine this number, Mr Dymond varied the orientation of the gun inside the flask, and for different scattering angles collected the electrons that suffered an energy loss equivalent to 20 to 55 volts; he constructed a series of curves of the angular distribution of these electrons for different values of the tension $\mathcal{V}$ applied to the electron gun. The angular distribution curve shows a very pronounced maximum for a low value of θ, and this maximum appears to approach $\theta = 0$ for increasing values of $\mathcal{V}$.

Another, less important, maximum appears towards $\theta = 50^\circ$ for a primary energy of about a hundred volts, and then moves for increasing values of $\mathcal{V}$ towards increasing θ. Finally, a very sharp maximum appears for a primary energy of about 200 volts at $\theta = 30^\circ$, and then seems independent of $\mathcal{V}$. These facts are summarised in the following table given by Dymond:[8]

$\mathcal{V}$ (volts)		Positions of the maxima (°)		
48.9	....	24	—	—
72.3	....	8	—	—
97.5	....	5	—	50
195	....	< 2.5	30	59
294	....	< 2.5	30	69
400	....	< 2.5	30	70

The above results must very probably be interpreted with the aid of the new Mechanics and are to be related to Mr Born's predictions. Nevertheless, as Mr Dymond very rightly says, 'the theoretical side of the problem is however not yet sufficiently advanced to give detailed information on the phenomena to be expected, so that the results above reported cannot be said to substantiate the wave mechanics except in the most general way'.[9]

12. *Experiments by Messrs C. Davisson and L. H. Germer.* — In 1923, Messrs Davisson and Kunsman [25] published peculiar results on the scattering of electrons at low speed. They directed a beam of electrons, accelerated by a potential difference of less than 1000 volts, onto a block of platinum at an incidence of 45° and determined the distribution of scattered electrons by collecting them in a Faraday cylinder. For potentials above 200 volts, one observed a steady decrease in scattering for increasing values of the deviation angle, but for smaller voltages

the curve of angular variation showed two maxima. By covering the platinum with a deposit of magnesium, one obtained a single small maximum for electrons of less than 150 volts. Messrs Davisson and Kunsman attributed the observed phenomena to the action of various layers of intra-atomic electrons on the incident electrons, but it seems rather, according to Elsasser's opinion, that the interpretation of these phenomena is a matter for the new Mechanics.

Resuming analogous experiments with Mr Germer [26], Mr Davisson obtained very important results this year, which appear to confirm the general predictions and even the formulas of Wave Mechanics.

The two American physicists sent homogeneous beams of electrons onto a crystal of nickel, cut following one of the 111 faces of the regular octahedron (nickel is a cubic crystal). The incidence being normal, the phenomenon necessarily had to show the ternary symmetry around the direction of the incident beam. In a cubic crystal cut in this manner, the face of entry is cut obliquely by three series of 111 planes, three series of 100 planes, and six series of 110 planes. If one takes as positive orientation of the normals to these series of planes the one forming an acute angle with the face of entry, then these normals, together with the direction of incidence, determine distinguished azimuths, which Messrs Davisson and Germer call azimuths (111), (100), (110), and for which they studied the scattering; because of the ternary symmetry, it evidently suffices to explore a single azimuth of each type.

Let us place ourselves at one of the distinguished azimuths and let us consider only the distribution of Ni atoms on the face of entry of the crystal, which we assume to be perfect. These atoms form lines perpendicular to the azimuth being considered and whose equidistance d is known from crystallographic data. The different directions of scattering being identified in the azimuthal plane by the angle θ of co-latitude, the waves scattered by the atoms in the face of entry must be in phase in directions such that one has

$$\theta = \arcsin\left(\frac{n\lambda}{d}\right) = \arcsin\left(\frac{n}{d}\frac{12.25}{\sqrt{\mathcal{V}}}\cdot 10^{-8}\right) \qquad (n \text{ integer}) \ .$$

One must then expect to observe maxima in these directions, for the scattering of the electrons by the crystal.

Now here is what Messrs Davisson and Germer observed. By gradually varying the voltage $\mathcal{V}$ that accelerates the electrons one observes, in the neighbourhood of *certain* values of $\mathcal{V}$, very distinct scattering maxima in directions whose co-latitude is accurately given by the above formula (provided one sets in general $n = 1$, and sometimes $n = 2$). There is direct numerical confirmation of the formulas of the new Dynamics; this is evidently a result of the highest importance.

However, the explanation of the phenomenon is not complete: one must explain why the scattering maxima are observed only in the neighbourhood of certain particular values of $\mathcal{V}$, and not for all values of $\mathcal{V}$. One interpretation naturally comes to mind: we assumed above that only the face of entry of the crystal played a role, but one can assume that the electron wave penetrates somewhat into the crystal and, further, in reality the face of entry will never be perfect and will be formed by several parallel 111 planes forming steps. In these conditions, it is not sufficient to consider the interference of the waves scattered by a single reticular plane at the surface, one must take into account the interference of the waves scattered by several parallel reticular planes. In order for there to be a strong scattering in a direction θ, θ and $\mathcal{V}$ must then satisfy not only the relation written above, but also another relation which is easy to find; the scattering must then be *selective*, that is to say, occur with [significant] intensity only for certain values of $\mathcal{V}$, as experiment shows. Of course, the theory that has just been outlined is a special case of Laue's general theory.

Unfortunately, as Messrs Davisson and Germer have themselves remarked, in order to obtain an exact prediction of the facts in this way, it is necessary to attribute to the separation of the 111 planes next to the face of entry a smaller value (of about 30%) than that provided by Crystallography and by direct measurements by means of X-rays. It is moreover not unreasonable to assume that the very superficial reticular planes have a spacing different from those of the deeper planes, and one can even try to connect this idea to our current conceptions concerning the equilibrium of crystalline gratings.

If one accepts the preceding hypothesis, the scattering must be produced by a very small number of reticular planes in the entirely superficial layer of the crystal; the concentration of electrons in preferred directions must then be much less pronounced than in the case of scattering by a whole unlimited spatial grating. Is it nevertheless sufficient in order to explain the 'peaks' observed by Davisson and Germer? To this question, Mr Patterson has recently provided an affirmative answer, by showing that the involvement of just two superficial reticular planes already suffices to predict exactly the variations of the selective reflection of electrons observed in the neighbourhood of

$$\theta = 50^\circ \ , \quad \mathcal{V} = 54 \text{ volts} \ .$$

To conclude, we can do no better than quote the conclusion of Mr Patterson [27]: 'The agreement of these results with calculation seems to indicate that the phenomenon can be explained as a diffraction of waves in the *outermost layers* of the crystal surface. It also appears [....] that a complete analysis of the results of such experiments will give valuable information as to the conditions prevailing in the actual surface, and that a new method has been made available for the investigation of the structure of crystals in a region which has up to the present almost completely escaped observation'.[10]

13. *Experiments by Messrs G. P. Thomson and A. Reid* [28]. — Very recently, Messrs Thomson and Reid have made the following results known: if a narrow pencil of homogeneous cathode rays passes normally through a celluloid film, and is then received on a photographic plate placed parallel to the film at 10 cm behind it, one observes rings around the central spot. With rays of 13 000 volts, a photometric examination has revealed the existence of three rings. By gradually increasing the energy of the electrons, one sees the rings appear around 2500 volts, and they have been observed up to 16 500 volts. The radii of the rings decrease when the energy increases and, it seems, approximately in inverse proportion to the speed, that is, to our wavelength λ.

These observations are very interesting, and again confirm the new conceptions in broad outline. Is it a question here of an atomic phenomenon analogous to those observed by Dymond, or else of a phenomenon of mutual interference falling into one of the categories studied by Debye and Scherrer, Hull, Debierne, Keesom and De Smedt? We are unable to say, and we limit ourselves to remarking that here the electrons used are relatively fast; this is interesting from the experimental point of view, because it is much easier to study electrons of a few thousand volts than electrons of about a hundred volts.

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[a] The style of the bibliography has been slightly modernised and uniformised with that used in the other reports (*eds.*).

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Discussion of Mr de Broglie's report

Mr Lorentz. — I should like to see clearly how, in the first form of your theory, you recovered Sommerfeld's quantisation conditions. You obtained a single condition, applicable only to the case where the orbit is closed: the wave must, after travelling along the orbit, finish in phase when it comes back to the initial point. But in most cases the trajectory is not closed; this happens, for example, for the hydrogen atom when one takes relativity into account; the trajectory is then a rosette, and never comes back to its initial point.

How did you find the quantisation conditions applicable to these multiperiodic problems?

Mr de Broglie. — The difficulty is resolved by considering pseudo-periods, as I pointed out in my Thesis (chap. III, p. 41). When a system is multiperiodic, with partial periods τ_1, τ_2, ..., τ_n, one can prove that one can find quasi-periods τ that are nearly exactly whole multiples of the partial periods:

$$\tau = m_1\tau_1 + \varepsilon_1 = m_2\tau_2 + \varepsilon_2 ... = m_n\tau_n + \varepsilon_n ,$$

the m_1, m_2, ..., m_n being integers and the ε_1, ε_2, ..., ε_n as small as one likes. The trajectory then never comes back to its initial point, but at the end of a quasi-period τ it comes back as closely as one likes to the initial position. One will then be led to write that, at the end of a quasi-period, the wave finishes in phase; now, there is an infinite number of quasi-periods, corresponding to all kinds of systems of values of the integers m_1, m_2, ..., m_n. In order that the wave finishes in phase after any one of these quasi-periods, it is necessary that one have[11]

$$\int_{\tau_1} p_1 dq_1 = n_1 h , \quad \int_{\tau_2} p_2 dq_2 = n_2 h , \quad ... , \quad \int_{\tau_n} p_n dq_n = n_n h ,$$

which gives exactly Sommerfeld's conditions.[a]

Mr Born. — The definition of the trajectory of a particle that Mr de Broglie has given seems to me to present difficulties in the case of a collision between an electron and an atom. In an elastic collision, the speed of the particle must be the same after the collision as before. I

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[a] Darrigol (1993, pp. 342–3, 364–5) shows that this derivation is faulty: the condition that the wave should finish in phase after any quasi-period does *not* imply the n separate conditions listed above.

should like to ask Mr de Broglie if that follows from his formula.

Mr de Broglie. — That follows from it, indeed.

Mr Brillouin. — It seems to me that no serious objection can be made to the point of view of L. de Broglie. Mr Born can doubt the real existence of the trajectories calculated by L. de Broglie, and assert that one will never be able to observe them, but he cannot prove to us that these trajectories do not exist. There is no contradiction between the point of view of L. de Broglie and that of other authors, since, in his report (§8, p. 388[12]) L. de Broglie shows us that his formulas[a] are in exact agreement with those of Gordon, at present accepted by all physicists.

Mr Pauli. — I should like to make a small remark on what seems to me to be the mathematical basis of Mr de Broglie's viewpoint concerning particles in motion on definite trajectories. His conception is based on the principle of conservation of charge:

$$\frac{\partial \rho}{\partial t} + \frac{\partial s_1}{\partial x} + \frac{\partial s_2}{\partial y} + \frac{\partial s_3}{\partial z} = 0 \quad \text{or} \quad \sum_{k=1}^{4} \frac{\partial s_k}{\partial x_k} = 0 \,, \tag{a}$$

which is a consequence of the wave equation, when one sets

$$i s_k = \psi \frac{\partial \psi^*}{\partial x_k} - \psi^* \frac{\partial \psi}{\partial x_k} + \frac{4\pi i}{h} \frac{e}{c} \Phi_k \psi \psi^* \,.$$

Mr de Broglie introduces, in place of the complex function ψ, the two real functions a and φ defined by

$$\psi = a e^{\frac{2\pi i}{h}\varphi} \,, \quad \psi^* = a e^{-\frac{2\pi i}{h}\varphi} \,.$$

Substituting these expressions into the expression for s_k yields:

$$s_k = \frac{4\pi}{h} a^2 \left(\frac{\partial \varphi}{\partial x_k} + \frac{e}{c} \Phi_k \right) \,.$$

From this follow the expressions given by Mr de Broglie for the velocity vector, defined by

$$v_1 = \frac{s_1}{\rho} \,, \quad v_2 = \frac{s_2}{\rho} \,, \quad v_3 = \frac{s_3}{\rho} \,. \tag{b}$$

Now if in a field theory there exists a conservation principle of the

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[a] That is, de Broglie's equations for the mean charge and current density to be used in semiclassical radiation theory (*eds.*).

form (a), it is always formally possible to introduce a velocity vector (b), depending on space and time, and to imagine furthermore corpuscles that move following the current lines of this vector. Something similar was already proposed in optics by Slater; according to him, light quanta should always move following the lines of the Poynting vector. Mr de Broglie now introduces an analogous representation for material particles.

In any case, I do not believe that this representation may be developed in a satisfactory manner; I intend to return to this during the general discussion.[a]

MR SCHRÖDINGER. — If I have properly understood Mr de Broglie, the *velocity* of the particles must have its analogue in a vector field composed of the three *spatial* components of the *current in a four-dimensional space*, after division of these by the component *with respect to time* (that is, the charge density). I should like simply to recall now that there exist still other vector quantities of a field, which can be made to correspond with the velocity of the particles, such as the components of the *momentum density* (see *Ann. d. Phys.*[13] **82**, 265). Which of the two analogies is the more convincing?

MR KRAMERS. — The fact that with independent particles in motion one cannot construct an energy-momentum tensor having the properties required by Maxwell's theory constitutes nevertheless a difficulty.

MR PAULI. — The quotient of the momentum by the energy density which Mr Schrödinger considers would in fact lead in a relativistic calculation to other particle trajectories than would the quotient of the densities of current and of charge.

MR LORENTZ. — In using his formulas for the velocity of the electron, has Mr de Broglie not calculated this velocity in particular cases, for example for the hydrogen atom?

MR DE BROGLIE. — When one applies the formula for the velocity to a wave function representing a stable state of the hydrogen atom according to Mr Schrödinger, one finds circular orbits. One does not

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[a] See pp. 509 ff. (*eds.*).

recover the elliptical orbits of the old theory (see my report, §8).

Mr Ehrenfest. — Can the speed of an electron in a stationary orbit be zero?

Mr de Broglie. — Yes, the speed of the electron can be zero.

Mr Schrödinger. — Mr de Broglie says that in the case of the hydrogen atom his hypothesis leads to *circular* orbits. That is true for the particular solutions of the wave equation that one obtains when one separates the problem in polar coordinates in space; perhaps it is still true for the solutions that one obtains by making use of parabolic or elliptical coordinates. But in the case of a degeneracy (as he considers it here) it is, in reality, not at all the particular solutions which have a significance, but only an *arbitrary* linear combination, with constant coefficients, of all the particular solutions belonging to the same eigenvalue, because there is no means of distinguishing between them, all linear combinations being equally justified in principle. In these conditions, much more complicated types of orbit will certainly appear. But I do not believe that in the atomic domain one may still speak of 'orbits'.

Mr Lorentz. — Does one know of such more complicated orbits?

Mr Schrödinger. — No, one does not know of them; but I simply wanted to say that if one finds circular orbits, that is due to a fortuitous choice of particular solutions that one considers, and this choice cannot be motivated in a way that has no arbitrariness.

Mr Brillouin. — Perhaps it is not superfluous to give some examples that illustrate well the meaning of Mr L. de Broglie's formulas, and that allow one to follow the motion of the particles guided by the phase wave. If the wave is plane and propagates freely, the trajectories of the particles are the rays normal to the wave surface. Let us suppose that the wave is reflected by a plane mirror, and let θ be the angle of incidence; the wave motion in front of the mirror is given by a superposition of the incident wave

$$\psi_1 = a_1 \cos 2\pi \left(\frac{t}{T} - \frac{x \sin\theta - z \cos\theta}{\lambda} \right)$$

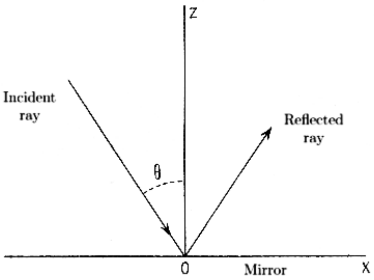

Fig. 1.

and the reflected wave

$$\psi_2 = a_1 \cos 2\pi \left(\frac{t}{T} - \frac{x \sin\theta + z \cos\theta}{\lambda} \right) ,$$

which gives

$$\psi = 2a_1 \cos \frac{2\pi z \cos\theta}{\lambda} \cos 2\pi \left(\frac{t}{T} - \frac{x \sin\theta}{\lambda} \right) .$$

This wave is put in L. de Broglie's canonical form

$$\psi = a \cos \frac{2\pi}{h} \varphi$$

with

$$a = 2a_1 \cos \frac{2\pi z \cos\theta}{\lambda} \quad \text{and} \quad \varphi = h \left(\frac{t}{T} - \frac{x \sin\theta}{\lambda} \right) .$$

Let us then apply L. de Broglie's formulas, in the simplified form given on page 384 (§5);[14] and let us suppose that it is a light wave guiding the photons; the velocity of these is

$$\vec{v} = -\frac{c^2}{h\nu} \overrightarrow{\text{grad}\,\varphi} .$$

We see that the projectiles move parallel to the mirror, with a speed $v_x = c \sin\theta$, less than c. Their energy remains equal to $h\nu$, because their mass has undergone a variation, according to the following formula (report by L. de Broglie, p. 383):[15]

$$M_0 = \sqrt{m_0^2 - \frac{h^2}{4\pi^2 c^2} \frac{\Box a}{a}} = \frac{h}{2\pi c} \sqrt{-\frac{\Box a}{a}} = \frac{h\nu}{c^2} \cos\theta .$$

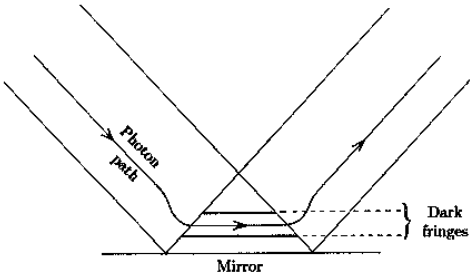

Fig. 2.

The mass of the photons, which is zero in the case where the wave propagates freely, is then assumed to take a non-zero value in the whole region where there is interference or deviation of the wave.

Let us draw a diagram for the case of a limited beam of light falling on a plane mirror; the interference is produced in the region of overlap of two beams. The trajectory of a photon will be as follows: at first a rectilinear path in the incident beam, then a bending at the edge of the interference zone, then a rectilinear path parallel to the mirror, with the photon travelling in a bright fringe and avoiding the dark interference fringes; then, when it comes out, the photon retreats following the direction of the reflected light beam.

No photon actually strikes the mirror, nevertheless the mirror suffers classical radiation pressure; it is in order to explain this fact that L. de Broglie assumes the existence of special stresses in the interference zone; these stresses, when added to the tensor of momentum flux transported by the photons, reproduce the classical Maxwell tensor; there is then no difference in the mechanical effects produced by the wave during its reflection by the mirror.

These remarks show how L. de Broglie's system of hypotheses preserves the classical formulas, and avoids a certain number of awkward paradoxes. One thus obtains, for example, the solution to a curious problem posed by G. N. Lewis (*Proc. Nat. Acad.* **12** (1926), 22 and 439), which was the subject of discussions between this author and R. C. Tolman and S. Smith (*Proc. Nat. Acad.* **12** (1926), 343 and 508).

Lewis assumed that the photons always follow the path of a light

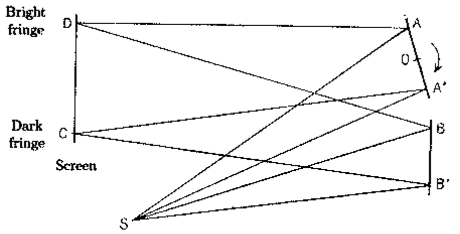

Fig. 3.

ray of geometrical optics, but that they choose, among the different rays, only those that lead from the luminous source to a bright fringe situated on an absorbing body. He then considered a source S whose light is reflected by two mirrors AA´ and BB´; the light beams overlap, producing interference [zones] in which one places a screen CD; the dimensions are assumed to be such that there is a bright fringe on one of the edges D of the screen and a dark fringe on the other edge C. Following the hypothesis of Lewis, the photons would follow only the paths SBD and SAD, which end at the bright fringe D; no photon will take the path SA´C or SB´C. All the photons come to strike the mirror AA´ on the edge A, so one could predict that this mirror would suffer a torque; if one made it movable around an axis O, it would tend to turn in the direction of the arrow.

This paradoxical conclusion is entirely avoided by L. de Broglie, since his system of hypotheses preserves the values of radiation pressure. This example shows clearly that there is a contradiction between the hypothesis of rectilinear paths for the photons (following the light rays) and the necessity of finding photons only where a bright interference fringe is produced, no photon going through the regions of dark fringes.

Mr Lorentz draws attention to a case where the classical theory and the photon hypothesis lead to different results concerning the ponderomotive forces produced by light. Let us consider reflection by the hypotenuse face of a glass prism, the angle of incidence being larger than the angle of total reflection. Let us place a[16] second prism behind the first, at a distance of the order of magnitude of the wavelength, or only a fraction of this length. Then, the reflection will no longer be total. The light waves that penetrate the layer of air reach the second prism before their intensity is too much weakened, and there give rise to a beam transmitted in the direction of the incident rays.

If, now, one calculates the Maxwell stresses on a plane situated in the layer of air and parallel to its surfaces, one finds that, if the angle of incidence exceeds a certain value (60° for example), there will be an *attraction* between the two prisms. Such an effect can never be produced by the motion of corpuscles, this motion always giving rise to a [positive] pressure as in the kinetic theory of gases.

What is more, in the classical theory one easily sees the origin of the 'negative pressure'. One can distinguish two cases, that where the electric oscillations are in the plane of incidence and that where this is so for the magnetic oscillations. If the incidence is very oblique, the oscillations of the incident beam that I have just mentioned are only slightly inclined with respect to the normal to the hypotenuse face, and the same is true for the corresponding oscillations in the layer of air.

One then has approximately, in the first case an electric field such as one finds between the electrodes of a capacitor, and in the second case a magnetic field such as exists between two opposite magnetic poles.

The effect would still remain if the second prism were replaced by a glass plate, but it must be very difficult to demonstrate this experimentally.

Notes to the translation

1 The integral sign is printed as '$\int_0$' in the original.

2 The French uses 'Wellenpacket' throughout.

3 Mis-spelt as 'Fuess'.

4 We follow the original English, which is a translation by Oppenheimer, from Born (1927, p. 355). De Broglie translates 'mechanics' as 'dynamique' and includes the words 'La Dynamique des Quanta' in the quotation, where Born has 'it' (referring to 'quantum mechanics').

5 The French reads 'ces' [these] rather than 'ses' [its].

6 'v' is misprinted as '$\mathcal{V}$'.

7 Consistently mis-spelt throughout the text as 'Elsaesser'.

8 For clarity, the presentation of the table has been slightly altered.

9 We have used the original English (Dymond 1927, p. 441).

10 We use here the original English text (Patterson 1927, p. 47). De Broglie changes 'of these results' to 'des résultats expérimentaux', omits the italics and translates 'valuable' by 'exacts'.

11 The last equality is misprinted as '$\int_{\tau_n} p_n dq = n_n h_n$'.

12 The original reads 'p. 18', which is presumably in reference to de Broglie's Gauthier-Villars 'preprint', in all likelihood circulated before the conference (preprints of other lectures were circulated as mimeographs). Cf. chapter 1, p. 19

13 '*Ann. de Phys.*' in the original.

14 This is 'p. 117' in the original.

15 Again, 'p. 117' in the original.

16 Misprinted as 'au' instead of 'un'.

Quantum mechanics[a]

BY MESSRS MAX BORN AND WERNER HEISENBERG

INTRODUCTION

Quantum mechanics is based on the intuition that the essential difference between atomic physics and classical physics is the occurrence of discontinuities (see in particular [1,4,58–63]).[b] Quantum mechanics should thus be considered a direct continuation of the quantum theory founded by Planck, Einstein and Bohr. Bohr in particular stressed repeatedly, already before the birth of quantum mechanics, that the discontinuities must lead to the introduction of new kinematical and mechanical concepts, so that indeed classical mechanics and its corresponding conceptual scheme should be abandoned [1,4]. Quantum mechanics tries to introduce the new concepts through a precise analysis of what is 'observable in principle'. In fact, this does not mean setting up the principle that a sharp division between 'observable' and 'unobservable' quantities is possible and necessary. As soon as a conceptual scheme is given, one can infer from the observations to other facts that are actually not observable directly, and the boundary between 'observable' and 'unobservable'[1] quantities becomes altogether indeterminate. But if the conceptual scheme itself is still unknown, it will be expedient to enquire only about the observations themselves, without drawing conclusions from them, because otherwise wrong concepts and prejudices taken over from before will block the way

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[a] Our translation follows the German typescript in AHQP-RDN, document M-0309. Discrepancies between the typescript and the published version are reported in the endnotes. The published version is reprinted in Heisenberg (1984, ser. B, vol. 2, pp. 58–99) (*eds.*).

[b] Numbers in square brackets refer to the bibliography at the end.

to recognising the physical relationships [Zusammenhänge]. At the same time the new conceptual scheme provides the anschaulich content of the new theory.[a] From a theory that is anschaulich in this sense, one can thus demand only that it is consistent in itself and that it allows one to predict unambiguously the results for all experiments conceivable in its domain. Quantum mechanics is meant as a theory that is in this sense anschaulich and complete for the micromechanical processes [46].[2]

Two kinds of discontinuities are characteristic of atomic physics: the existence of corpuscles (electrons, light quanta) on the one hand, and the occurrence of discrete stationary states (discrete[3] energy values, momentum values etc.) on the other. Both kinds of discontinuities can be introduced in the classical theory only through artificial auxiliary assumptions. For quantum mechanics, the existence of discrete stationary states and energy values is just as natural as the existence of discrete eigenoscillations in a classical oscillation problem [4]. The existence of corpuscles will perhaps later turn out to be reducible just as easily to discrete stationary states of the wave processes (quantisation of the electromagnetic waves on the one hand, and of the de Broglie waves on the other) [4], [54].

The discontinuities, as the notion of 'transition probabilities' already shows, introduce a statistical element into atomic physics. This statistical element forms an *essential* part of the foundations of quantum mechanics (see in particular [4,30,38,39,46,60,61,62]);[4] according to the latter, for instance, in many cases the course of an experiment is determinable from the initial conditions only statistically, at least if in fixing the initial conditions one takes into account only the experiments conceivable in principle up to now. This consequence of quantum mechanics is empirically testable. Despite its statistical character, the theory nevertheless accounts for the apparently fully causal determination of macroscopic[5] processes. In particular, the principles of conservation of energy and momentum hold exactly also in quantum mechanics. There seems thus to be no empirical argument against accepting fundamental indeterminism for the microcosm.

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[a] For the notion of Anschaulichkeit, see the comments in sections 3.4.7, 4.6 and 8.3 (*eds.*).

I. — THE MATHEMATICAL METHODS OF QUANTUM MECHANICS[a]

The phenomenon for whose study the mathematical formalism of quantum mechanics was first developed is the spontaneous radiation of an excited atom. After innumerable attempts to explain the structure of the line spectra with classical mechanical models had proved inadequate, one returned to the direct description of the phenomenon on the basis of its simplest empirical laws (Heisenberg [1]). First among these is Ritz's combination principle, according to which the frequency of each spectral line of an atom appears as the difference of two terms $\nu_{ik} = T_i - T_k$; thus the set of all lines of the atom will be best described by specifying a quadratic array [Schema], and since each line possesses besides its frequency also an intensity and a phase, one will write in each position of the array an elementary oscillation function with complex amplitude:

$$\begin{pmatrix} q_{11}e^{2\pi i\nu_{11}t} & q_{12}e^{2\pi i\nu_{12}t} & \dots \\ q_{21}e^{2\pi i\nu_{21}t} & q_{22}e^{2\pi i\nu_{22}t} & \dots \\ \dots\dots\dots & \dots\dots\dots & \dots \end{pmatrix} . \tag{1}$$

This array is understood as representing a coordinate q as a function of time in a similar way as the totality of terms of the Fourier series

$$q(t) = \sum_n q_n e^{2\pi i\nu_n t}, \quad \nu_n = n\nu_0$$

in the classical theory; except that now because of the two indices the sum no longer makes sense. The question arises of which expressions correspond to functions of the classical coordinate, for instance to the square q^2. Now, such arrays ordered by two indices occur as *matrices* in mathematics in the theory of quadratic forms and of linear transformations; the composition of two linear transformations,

$$x_k = \sum_l a_{kl} y_l \ , \qquad y_l = \sum_j b_{lj} z_j \ ,$$

to form a new one,

$$x_k = \sum_j c_{kj} z_j \ ,$$

then corresponds to the composition or multiplication of the matrices

$$ab = c, \quad \text{that is,} \quad \sum_l a_{kl} b_{lj} = c_{kj} \ . \tag{2}$$

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[a] Section 3.3 contains additional material on the less familiar aspects of the formalisms presented here (*eds.*).

This multiplication in general is *not* commutative. It is natural to apply this recipe to the array of the atomic oscillations (Born and Jordan [2], Dirac [3]); it is immediately evident that because of Ritz's formula $\nu_{ik} = T_i - T_k$ no new frequencies appear, just as in the classical theory in the multiplication of two Fourier series, and herein lies the first justification for the procedure. By repeated application of additions and multiplications one can define arbitrary matrix functions.

The analogy with the classical theory leads further to allowing as representatives of real quantities only those matrices that are 'Hermitian', that is, whose elements go over to the complex conjugate numbers under permutation of the indices. The discontinuous nature of the atomic processes here is put into the theory from the start as empirically established. However, this does not establish yet the connection with quantum theory and its characteristic constant h. This is also achieved, by carrying over the content[6] of the Bohr-Sommerfeld quantum conditions in a form given by Kuhn and Thomas, in which they are written as relations between the Fourier coefficients of the coordinates q and momenta p. In this way one obtains the matrix equation

$$pq - qp = \frac{h}{2\pi i} \cdot 1 \ , \tag{3}$$

where 1 means the unit matrix. The matrix p thus does not 'commute' with q. For several degrees of freedom the commutation relation (3) holds for every pair of conjugate quantities, while the q_k commute with each other, the p_k with each other, and also the p_k with the non-corresponding q_k.

In order to construct the new mechanics (Born, Heisenberg and Jordan [4]), one carries over as far as possible the notions of the classical theory. It is possible to define the differentiation of a matrix with respect to time and that of a matrix function with respect to an argument matrix. One can thus carry over to the matrix theory the canonical equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \ , \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \ ,$$

where one should understand $H(p, q)$ as the same function of the matrices p, q that occurs in the classical theory as a function of the numbers p, q. (To be sure,[7] ambiguity can occur because of the noncommutativity of the multiplication; for example, p^2q is different from pqp.) This procedure was tested in simple examples (harmonic and anharmonic oscillator). Further, one can prove the theorem of conservation of energy, which for non-degenerate systems (all terms T_k different from each other, or: all frequencies ν_{ik} different from zero) here takes the form: for the solutions p, q of the canonical equations the Hamiltonian function $H(p,q)$ becomes a diagonal matrix W. It follows immediately that the elements of this diagonal matrix represent the terms T_n of Ritz's formula multiplied by h (Bohr's frequency condition). It is particularly important to realise that conversely the requirement

$$H(p,q) = W \qquad \text{(diagonal matrix)}$$

is a complete substitute for the canonical equations of motion, and leads to unambiguously determined solutions even if one allows for degeneracies (equality of terms, vanishing frequencies).

By a matrix with elements that are harmonic functions of time, one can of course represent only quantities (coordinates) that correspond to time-periodic quantities of the classical theory. Therefore cyclic coordinates (angles), which increase proportionally to time, cannot be treated at present.[a] Nevertheless, one easily manages to subject rotating systems to the matrix method by representing the Cartesian components of the angular momentum with matrices [4].[8] One obtains thereby expressions for the energy[b] that differ characteristically from the corresponding classical ones; for instance the modulus[9] of the total angular momentum is not equal to $\frac{h}{2\pi}j$ $(j = 0, 1, 2, \ldots)$, but to $\frac{h}{2\pi}\sqrt{j(j+1)}$, in accordance with empirical rules that Landé and others had derived from the term splitting in the Zeeman effect.[c] Further, one obtains for the changes in the angular quantum numbers [Rotationsquantenzahlen] the correct selection rules and intensity formulas, as had already been arrived at earlier by correspondence arguments and confirmed by the Utrecht observations.[d]

Pauli [6], avoiding angular variables, even managed to work out the hydrogen atom with matrix mechanics, at least with regard to the energy values and some aspects of the intensities.

Asking for the most general coordinates for which the quantum mechanical laws are valid leads to the generalisation of the notions of canonical variables and canonical transformations known from the classical theory. Dirac [3] has noted that the content of the expressions such as

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[a] This point is taken up again shortly after eq. (10). (*eds.*).

[b] Angular momentum is of course responsible for a characteristic splitting of the energy terms (*eds.*).

[c] For Landé's work on the anomalous Zeeman effect, see Mehra and Rechenberg (1982a, sec. IV.4, esp. pp. 467–76 and 482–5) (*eds.*).

[d] For the 'Utrecht observations' see Mehra and Rechenberg (1982a, sec. VI.6, pp. 647–8) and Mehra and Rechenberg (1982b, sec. III.4, esp. pp. 154–61) (*eds.*).

$\frac{2\pi i}{h}(p_k q_l - q_l p_k) - \delta_{kl}$, which appear in the commutation relations of the type (3) corresponds^{10} to that of the Poisson brackets, whose vanishing in classical mechanics characterises a system of variables as canonical. Therefore also in quantum mechanics one will denote as canonical every system of matrix pairs p, q that satisfy the commutation relations, and as a canonical transformation every transformation that leaves these relations invariant. One can write these with the help of an arbitrary matrix S in the form[a]

$$P = S^{-1} p S, \quad Q = S^{-1} q S \ , \tag{4}$$

and in a certain sense this is the most general canonical transformation. Then for an arbitrary function one has

$$f(P, Q) = S^{-1} f(p, q) S \ .$$

Now one can also carry over the main idea of the Hamilton-Jacobi theory [4]. Indeed, if the Hamiltonian function H is given as a function of any known canonical matrices p_0, q_0, then the solution of the mechanical problem defined by H reduces to finding a matrix S that satisfies the equation

$$S^{-1} H(p_0, q_0) S = W \ . \tag{5}$$

This is an analogue of the Hamilton-Jacobi differential equation of classical mechanics.

Exactly as there, also here perturbation theory can be treated most clearly with the help of equation (5). If H is given as a power series in some small parameter

$$H = H_0 + \lambda H_1 + \lambda^2 H_2 + \ldots$$

and the mechanical problem is solved for $\lambda = 0$, that is, $H_0 = W_0$ is known as a diagonal matrix, then the solution to (5) can be obtained easily as a power series

$$S = 1 + \lambda S_1 + \lambda^2 S_2 + \ldots$$

by successive approximations. Among the numerous applications of this procedure, only the derivation of Kramers' dispersion formula shall be mentioned here, which results if one assumes that the light-emitting and the scattering systems are weakly coupled and if one calculates the perturbation on the latter ignoring the backreaction [4].[a]

[a] Cf. p. 99 above (*eds.*).

[a] In other words, one considers just the scattering system under an external

The theory of the canonical transformations leads to a deeper conception, which later became essential in understanding the physical meaning of the formalism.

To each matrix $a = (a_{nm})$ one can associate a quadratic (more precisely: Hermitian) form[a]

$$\sum_{nm} a_{nm}\varphi_n\bar{\varphi}_m$$

of a sequence of variables $\varphi_1, \varphi_2 \ldots$, or also a linear transformation of the sequence of variables $\varphi_1, \varphi_2 \ldots$, into another one $\psi_1, \psi_2 \ldots$[11]

$$\psi_n = \sum_m a_{nm}\varphi_m \ , \tag{6}$$

where provisionally the meaning of the variables φ_n and ψ_n shall be left unspecified; we shall return to this.

A transformation (6) is called 'orthogonal' if it maps the identity form into itself

$$\sum_n \varphi_n\bar{\varphi}_n = \sum_n \psi_n\bar{\psi}_n \ . \tag{7}$$

Now these orthogonal transformations of the auxiliary variables φ_n immediately turn out to be essentially identical to the canonical transformations of the q and p matrices; the Hermitian character and the commutation relations are preserved. Further, one can replace the matrix equation (5) by the equivalent requirement [4]: the form

$$\sum_{nm} H_{nm}(q_0, p_0)\varphi_n\bar{\varphi}_m$$

is to be transformed orthogonally into a sum of squares

$$\sum_n w_n\psi_n\bar{\psi}_n \ . \tag{8}$$

The fundamental problem of mechanics is thus none other than the principal axes problem for surfaces of second order in infinite-dimensional space, occurring everywhere in pure and applied mathematics and variously studied. As is well known, this is equivalent to asking for the values of the parameter W for which the linear equations

$$W\varphi_n = \sum_m H_{nm}\varphi_m \tag{9}$$

perturbation (Born, Heisenberg and Jotrdan [4], section 2.4, in particular eq. (32)). See also Mehra and Rechenberg (1982c, ch. III, esp. pp. 93–4 and 103–9) (*eds.*).

[a] $\bar{\varphi}$ denotes the complex number conjugate to φ.

have a non-identically vanishing solution. The values $W = W_1, W_2, \ldots$ are called eigenvalues of the form H; they are the energy values (terms) of the mechanical system. To each eigenvalue W_n corresponds an eigensolution $\varphi_k = \varphi_{kn}$.[a] The set of these eigensolutions evidently again forms a matrix and it is easy to see that this is identical with the transformation matrix S appearing in (5).[b]

The eigenvalues, as is well known, are invariant under orthogonal transformations of the φ_k,[12] and since these correspond to the canonical substitutions of the p and q matrices, one recognises immediately the canonical invariance of the energy values W_n.

While the quantum theoretical matrices do not belong to the class of matrices (finite and bounded infinite[13]) investigated by the mathematicians (especially by Hilbert and his school), one can nevertheless carry over the main aspects of the known theory to the more general case. The precise formulation of these theorems[14] has been recently given by J. von Neumann [42] in a paper to which we shall have to return.[15]

The most important result that is achieved in this way is the theorem that a form cannot always be decomposed into a sum of squares (8), but that there also occur invariant integral components

$$\int W\psi(W)\overline{\psi(W)}dW \ , \tag{10}$$

where the sequence of variables $\psi_1, \psi_2, \ldots$ has to be complemented by the continuous distribution $\psi(W)$.

In this way the continuous spectra appear in the theory in the most natural way. But this implies by no means that in this domain the classical theory comes again into its own. Also here the characteristic discontinuities of quantum theory remain; also in the continuous spectrum a (spontaneous) state transition consists of a 'jump' of the system from a point W' to another one W'' with emission of a wave $q(W, W')e^{2\pi i\nu t}$ with the frequency $\nu = \frac{1}{h}(W' - W'')$.

The main defect of matrix mechanics consists in its clumsiness, even helplessness, in the treatment of non-periodic quantities, such as angular variables or coordinates that attain infinitely large values (e.g. hyperbolic trajectories). To overcome this difficulty two essentially different routes have been taken, the operator calculus of Born and Wiener [21], and the so-called[16] q-number theory of Dirac [7].

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[a] This is the notation used by Born and Heisenberg: the nth eigensolution is represented by an infinite vector with components labelled by k (*eds.*).

[b] This point is made more explicit after eq. (17). See also the relevant contributions by Dirac and by Kramers in the general discussion, pp. 491 and 495 (*eds.*).

The latter starts from the idea that a great part of the matrix relations can be obtained without an explicit representation of the matrices, simply on the basis of the rules for operating with the matrix symbols. These depart from the rules for numbers only in that the multiplication is generally not commutative. Dirac therefore considers abstract quantities, which he calls q-numbers (as opposed to the ordinary c-numbers) and with which he operates according to the rules of the noncommutative algebra. It is therefore a kind of hypercomplex number system. The commutation relations are of course preserved. The theory acquires an extraordinary resemblance to the classical one; for instance, one can introduce angle and action variables w, J and expand any q-number into a Fourier series with respect to the w; the coefficients are functions of the J and turn out to be identical to the matrix elements if one replaces the J by integer multiples of h. By his method Dirac has achieved important results, for instance worked out the hydrogen atom independently of Pauli [7] and determined the intensity of radiation in the Compton effect [12]. A drawback of this formalism — apart from the quite tiresome dealing with the noncommutative algebra — is the necessity to replace at a certain point of the calculation certain q-numbers with ordinary numbers (e.g. $J = hn$), in order to obtain results comparable with experiment. Special 'quantum conditions' which had disappeared from matrix mechanics are thus needed again.

The operator calculus differs from the q-number method in that it does not introduce abstract hypercomplex numbers, but concrete, constructible mathematical objects that obey the same laws, namely operators or functions in the space of infinitely many variables. The method is by Eckart [22] and was then developed further by many others following on from Schrödinger's wave mechanics, especially by Dirac [38] and Jordan [39] and in an impeccable mathematical form by J. von Neumann [42]; it rests roughly on the following idea.

A sequence of variables $\varphi_1, \varphi_2, \ldots$ can be interpreted as a point in an infinite-dimensional space. If the sum of squares $\sum_n |\varphi_n|^2$ converges, then it represents a measure of distance, a Euclidean metric [Massbestimmung], in this space; this metric space of infinitely many dimensions is called for short a Hilbert space. The canonical transformations of matrix mechanics correspond thus to the rotations of the Hilbert space. Now, however, one can also fix a point in this space other than by the specification of discrete coordinates $\varphi_1, \varphi_2, \ldots$. Take for instance a complete, normalised orthogonal system of functions $f_1(q), f_2(q), \ldots$, that is one for which[17]

$$\int f_n(q)\overline{f_m(q)}dq = \delta_{nm} = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \ ; \end{cases} \tag{11}$$

the variable q can range here over an arbitrary, also multi-dimensional domain. If one then sets (Lanczos [23])[18]

$$\begin{cases} \varphi(q) & = \sum_n \varphi_n f_n(q) \\ H(q', q'') & = \sum_{nm} H_{nm} f_n(q')\overline{f_m(q'')} \ , \end{cases} \tag{12}$$

the linear equations (9)[19] turn into the integral equation[20]

$$W\varphi(q') = \int H(q', q'')\varphi(q'')dq'' \ . \tag{13}$$

This relation established through (12) means thus nothing but a change of the coordinate system in the Hilbert space, given by the orthogonal transformation matrix $f_n(q)$ with one discrete and one continuous index.

One sees thus that the preference for ‘discrete’ coordinate systems in the original version of the matrix theory is by no means something essential. One can just as well use ‘continuous matrices’ such as $H(q', q'')$. Indeed, the specific representation of a point in the Hilbert space by projection onto certain orthogonal coordinate axes does not matter at all; rather, one can summarise equations (9) and (13) in the more general equation

$$W\varphi = H\varphi \ , \tag{14}$$

where H denotes a linear operator which transforms the point φ of the Hilbert space into another. The equation requires to find those points φ which under the operation H only suffer a displacement along the line joining them to the origin.[21] The points satisfying this condition determine an orthogonal system of axes, the principal axes frame of the operator H; the number of axes is finite or infinite, in the latter case distributed discretely or continuously, and the eigenvalues W are the lengths of the principal axes. The linear operators in the Hilbert space are thus the general concept that can serve to represent a physical quantity mathematically. The calculus with operators proceeds obviously according to the same rules as the one with Dirac’s q-numbers; they[22] constitute a realisation of this abstract notion. So far we have analysed the situation with the example of the Hamiltonian function, but the same holds for any quantum mechanical quantity. Any coordinate q can be written, instead of as a matrix with discrete indices q_{nm}, also as a function of two continuous variables $q(q', q'')$ by projection onto an orthogonal system of functions, or, more generally, can also be considered as a linear operator in the Hilbert space; then it has eigenvalues that are invariant, and eigensolutions with respect to each orthogonal coordinate system. The same holds for a momentum p and every function of q and p, indeed for every quantum mechanical 'quantity'. While in the classical theory physical quantities are represented by variables that can take numerical values from an arbitrary value range, a physical quantity in quantum theory is represented by a linear operator and the stock of values that it can take by the eigenvalues of the corresponding principal axes problem in the Hilbert space.

In this view, Schrödinger's wave mechanics [24] appears formally as a special case. The simplest operator whose characteristic values are all the real numbers, is in fact the multiplication of a function $F(q)$ by the real number q; one writes it simply q. Then, however, the eigenfunctions are 'improper' functions; for according to (14) they must have the property of being everywhere zero except if $W = q$. Dirac [38] has introduced for the representation of such improper functions the 'unit function' $\delta(s)$, which should always be zero when $s \neq 0$, but for which nonetheless $\int_{-\infty}^{+\infty} \delta(s) ds = 1$ should hold. Then one can write down the (normalised) eigenfunctions

$$\varphi(q, W) = q\delta(W - q) \tag{15}$$

belonging to the operator q.

The conjugate to the operator q is the differential operator

$$p = \frac{h}{2\pi i}\frac{\partial}{\partial q} \ ; \tag{16}$$

indeed, the commutation relation (3) holds, which means just the trivial identity

$$(pq - qp)\mathcal{F}(q) = \frac{h}{2\pi i}\left\{\frac{d}{dq}(q\mathcal{F}) - q\frac{d\mathcal{F}}{dq}\right\} = \frac{h}{2\pi i}\mathcal{F}(q) \ .$$

If one now constructs a Hamiltonian function out of p, q (or out of several such conjugate pairs), then equation (14) becomes a differential equation for the quantity $\varphi(q)$:

$$H(q, \frac{h}{2\pi i}\frac{\partial}{\partial q})\varphi(q) = W\varphi(q) \ . \tag{17}$$

This is Schrödinger's wave equation, which appears here as a special case of the operator theory. The most important point about this formulation of the quantum laws (apart from the great advantage of connecting to known mathematical methods) is the replacement of all 'quantum conditions', such as were still necessary in Dirac's theory of q-numbers, by the simple requirement that the eigenfunction $\varphi(q) = \varphi(q, W)$ should be everywhere finite in the domain of definition of the variables q; from this, in the event, a discontinuous spectrum of eigenvalues W_n (along with continuous ones) arises automatically. But Schrödinger's eigenfunction $\varphi(q, W)$ is actually nothing but the transformation matrix S of equation (5), which one can indeed also write in the form

$$HS = SW \ ,$$

analogous to (17).

Dirac [38] has made this state of affairs even clearer by writing the operators q and p and thereby also H as integral operators, as in (13); then one has to set

$$\begin{aligned} q\mathcal{F}(q') &= \int q''\delta(q'-q'')\mathcal{F}(q'')dq'' &&= q'\mathcal{F}(q') \ , \\ p\mathcal{F}(q') &= \int \frac{h}{2\pi i}\delta'(q'-q'')\mathcal{F}(q'')dq'' &&= \frac{h}{2\pi i}\frac{d\mathcal{F}}{dq'} \ , \end{aligned} \tag{18}$$

where, however, the occurrence of the derivative of the singular function δ has to be taken into the bargain. Then Schrödinger's equation (17) takes the form (13).

The direct passage to the matrix representation in the strict sense takes place by inverting the formulas (12), in which one identifies the orthogonal system $f_n(q)$ with the eigenfunctions $\varphi(q, W_n)$ belonging to the discrete spectrum. If T is an arbitrary operator (constructed from q and $p = \frac{h}{2\pi i}\frac{\partial}{\partial q}$), define the corresponding matrix T_{nm} by the coefficients of the expansion

$$T\varphi_n(q) = \sum_m T_{nm}\varphi_m(q) \tag{19}$$

or

$$T_{nm} = \int \overline{\varphi_m(q)}T\varphi_n(q)dq \ ; \tag{19a}$$

then one easily sees that equation (17) is equivalent to (9).

The further development of the formal theory has taken place in close connection with its physical interpretation, to which we therefore turn first.

II. — PHYSICAL INTERPRETATION

The most noticeable defect of the original matrix mechanics consists in the fact that at first it appears to give information not about actual phenomena, but rather only about possible states and processes. It allows one to calculate the possible stationary states of a system; further it makes a statement about the nature of the harmonic oscillation that can manifest itself as a light wave in a quantum jump. But it says nothing about when a given state is present, or when a change is to be expected. The reason for this is clear: matrix mechanics deals only with closed periodic systems, and in these there are indeed no changes. In order to have true processes,[23] as long as one remains in the domain of matrix mechanics, one must direct one's attention to a *part* of the system; this is no longer closed and enters into interaction with the rest of the system. The question is what matrix mechanics can tell us about this.

Imagine, for instance, two systems 1 and 2 weakly coupled to each other (Heisenberg [35], Jordan [36]).[a] For the total system conservation of energy then holds; that is, H is a diagonal matrix. But for a subsystem, for instance 1, $H^{(1)}$ is not constant, the matrix has elements off the diagonal.[24] The energy exchange can now be interpreted in two ways: for one, the periodic elements of the matrix of $H^{(1)}$ (or of $H^{(2)}$) represent a slow beating, a continuous oscillation of the energy to and fro; but at the same time, one can also describe the process with the concepts of the discontinuum theory and say that system 1 performs quantum jumps and carries over the energy that is thereby freed to system 2 as quanta, and vice versa. But one can now show that these two apparently very different views do not contradict each other at all. This rests on a mathematical theorem that states the following:

Let $f(W_n^{(1)})$ be any function of the energy values $W_n^{(1)}$ of the isolated subsystem 1; if one forms the same function of the matrix $H^{(1)}$ that represents the energy of system 1 in the presence of the coupling to system 2, then $f(H^{(1)})$ is a matrix that does not consist only of diagonal elements $f(H^{(1)})_{nn}$. But these represent the time-averaged value of the quantity $f(H^{(1)})$. The effect of the coupling is thus measured by the difference[25]

$$\delta\overline{f_n} = f(H^{(1)})_{nn} - f(W_n^{(1)}) \ .$$

The first part of the said theorem now states that $\delta\overline{f_n}$ can be brought

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[a] The form of the result as given here is similar to that in Heisenberg [35]. For further details, see the discussion in section 3.4.4 (*eds.*).

into the form[26]

$$\delta\overline{f_n} = \sum_m \{f(W_m) - f(W_n)\}\Phi_{nm} \ . \tag{20}$$

This can be interpreted thus: the time average of the change in f due to the coupling is the arithmetic mean, with certain weightings Φ_{nm}, of all possible jumps of f for the isolated system.

These Φ_{nm} will have to be called 'transition probabilities'. The second part of the theorem determines the Φ_{nm} through the features of the coupling. Namely, if $p_1^0, q_1^0, p_2^0, q_2^0$ are coordinates satisfying the evolution equations of the uncoupled systems, for which therefore $H^{(1)}$ and $H^{(2)}$ on their own are diagonal matrices, one can then think of the energy, including the interaction, as expressed as a function of these quantities. Then the solution of the mechanical problem according to (5)[27] reduces to constructing a matrix S that satisfies the equation

$$S^{-1}H(p_1^0, q_1^0, p_2^0, q_2^0)S = W \ .$$

Denoting the states of system 1 by n_1, those of system 2 by n_2, a state of the total system is given by $n_1 n_2$,[28] and to each transition $n_1 n_2 \rightarrow m_1 m_2$ corresponds an element of S, $S_{n_1 n_2, m_1 m_2}$. Then the result is:[29]

$$\Phi_{n_1 m_1} = \sum_{n_2 m_2} |S_{n_1 n_2, m_1 m_2}|^2 \ . \tag{21}$$

The squares of the elements of the S-matrix thus determine the transition probabilities. The individual sum term $|S_{n_1 n_2, m_1 m_2}|^2$ in (21) obviously means that component of the transition probability for the jump $n_1 \rightarrow m_1$ of system 1 that is induced by the jump $n_2 \rightarrow m_2$ of system 2.

By means of these results the contradiction between the two views from which we started is removed. Indeed, for the mean values, which alone may be observed, the conception of continuous beating always leads to the same result as the conception of quantum jumps.

If one asks the question *when* a quantum jump occurs, the theory provides no answer. At first it seemed as if there were a gap here which might be filled with further probing. But soon it became apparent that this is not so, rather, that it is a failure of principle, which is deeply anchored in the nature of the possibility of physical knowledge [physikalisches Erkenntnisvermögen].

One sees that quantum mechanics yields mean values correctly, but cannot predict the occurrence of an individual event. Thus determinism, held so far to be the foundation of the exact natural sciences, appears here to go no longer unchallenged. Each further advance in the interpretation of the formulas has shown that the system of quantum mechanical formulas can be interpreted consistently only from the point of view of a fundamental indeterminism, but also, at the same time, that the totality of empirically ascertainable facts can be reproduced by the system of the theory.

In fact, almost all observations in the field of atomic physics have a statistical character; they are countings, for instance of atoms in a certain state. While the determinateness of an individual process is assumed by classical physics, in fact it plays practically no role, because the microcoordinates that exactly determine an atomic process can never all be given; therefore by averaging they are eliminated from the formulas, which thereby become statistical statements. It has become apparent that quantum mechanics represents a merging of mechanics and statistics, in which the unobservable microcoordinates are eliminated.

The clumsiness of the matrix theory in the description of processes developing in time can be avoided by making use of the more general formalisms[30] we have described above. In the general equation (14) one can easily introduce time explicitly by invoking the theorem of classical mechanics that energy W and time t behave as canonically conjugate quantities; in quantum mechanics it corresponds to having a commutation relation

$$Wt - tW = \frac{h}{2\pi i} \ .$$

Thus for W one can posit the operator $\frac{h}{2\pi i}\frac{\partial}{\partial t}$. Equation (14) then reads

$$\frac{h}{2\pi i}\frac{\partial \varphi}{\partial t} = H\varphi \ , \tag{22}$$

and here one can consider H as depending explicitly on time. A special case of this is the equation

$$\left\{H(q, \frac{h}{2\pi i}\frac{\partial}{\partial q}) - \frac{h}{2\pi i}\frac{\partial}{\partial t}\right\}\varphi(q) = 0 \ , \tag{22a}$$

given by Schrödinger [24],[31] which stands to (17) in the same relation as (22) to (14), as well as the form:

$$\frac{h}{2\pi i}\frac{\partial \varphi(q')}{\partial t} = \int H(q', q'')\varphi(q'')dq'' \ , \tag{22b}$$

much used by Dirac, which relates to the integral formula (13). Essentially, the introduction of time as a numerical variable reduces to thinking of the system under consideration as coupled to another one and neglecting the reaction on the latter. But this formalism is very convenient and leads to a further development of the statistical view,[a] namely, if one considers the case where an explicitly time-dependent perturbation $V(t)$ is added to a time-independent energy function H^0, so that one has the equation

$$\frac{h}{2\pi i}\frac{\partial\varphi}{\partial t} = \left\{H^0 + V(t)\right\}\varphi \tag{23}$$

(Dirac [37], Born [34]).[32] Now if φ_n^0 are the eigenfunctions of the operator H^0, which for the sake of simplicity we assume to be discrete, the desired quantity φ can be expanded in terms of these:

$$\varphi(t) = \sum_n c_n(t)\varphi_n^0 \ . \tag{24}$$

The $c_n(t)$ are then the coordinates of φ in the Hilbert space with respect to the orthogonal system φ_n^0; they can be calculated from the differential equation (23), if their initial values $c_n(0)$ are given. The result can be expressed as:

$$c_n(t) = \sum_m S_{nm}(t)c_m(0) \ , \tag{25}$$

where $S_{nm}(t)$ is an orthogonal matrix depending[33] on t and determined by $V(t)$.

The temporal process is thus represented by a rotation of the Hilbert space or by a canonical transformation (4) with the time-dependent matrix S.

Now how is one to interpret this?

From the point of view of Bohr's theory a system can always be in only *one* quantum state. To each of these belongs an eigensolution φ_n^0 of the unperturbed system. If now one wishes to calculate what happens to a system that is initially in a certain state, say the kth, one has to choose $\varphi = \varphi_k^0$ as the initial condition for equation (23), i.e. $c_n(0) = 0$ for $n \neq k$, and $c_k(0) = 1$. But then, after the perturbation is over, $c_n(t)$ will have become equal to $S_{nk}(t)$, and the solution consists of a superposition of eigensolutions. According to Bohr's principles it makes no sense to say a system is simultaneously in several states. The only possible interpretation seems to be statistical: the superposition of several eigensolutions expresses that through the perturbation the initial state can go over to any other quantum state, and it is clear that as measure for the transition probability one has to take the quantity

$$\Phi_{nk} = |S_{nk}(t)|^2 \ ;$$

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[a] See the discussion in section 3.4 (*eds.*).

because then one obtains again equation (20) for the average change of any state function.

This interpretation is supported by the fact that one establishes the validity of Ehrenfest's adiabatic theorem (Born [34]); one can show that under an infinitely slow action, one has

$$\Phi_{nn} \to 1, \qquad \Phi_{nk} \to 0 \qquad (n \neq k) \ ,$$

that is, the probability of a jump tends to zero.

But this assumption also leads immediately to an interpretation of the $c_n(t)$ themselves: the $|c_n(t)|^2$ must be the state probabilities [Zustandswahrscheinlichkeiten].

Here, however, one runs into a difficulty of principle that is of great importance, as soon as one starts from an initial state for which not all the $c_n(0)$ except one vanish. Physically, this case occurs if a system is given for which one does not know exactly the quantum state in which it is, but knows only the probability $|c_n(0)|^2$ for each quantum state. As a matter of fact, the phases [Arcus] of the complex quantities $c_n(0)$ still remain indefinite; if one sets $c_n(0) = |c_n(0)|e^{i\gamma_n}$, then the γ_n denote some phases whose meaning needs to be established. The probability distribution at the end of the perturbation according to (25) is then

$$|c_n(t)|^2 = \Big| \sum_m S_{nm}(t) c_m(0) \Big|^2 \tag{26}$$

and not

$$\sum_m |S_{nm}(t)|^2 |c_m(0)|^2 \ , \tag{27}$$

as one might suppose from the usual probability calculus.

Formula (26), following Pauli, can be called the theorem of the *interference of probabilities*; its deeper meaning has become clear only through the wave mechanics of de Broglie and Schrödinger, which we shall presently discuss. Before this, however, it should be noted that this 'interference' does not represent a contradiction with the rules of the probability calculus, that is, with the assumption that the $|S_{nk}|^2$ are quite usual probabilities.[a] In fact, the composition rule (27) follows from the concept of probability for the problem treated here when and only when the relative number, that is, the probability $|c_n|^2$ of the atoms in the state n, has been *established* beforehand *experimentally.*[34] In this

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[a] The notation $|S_{nm}|^2$ would probably be clearer, at least according to the reading of this passage proposed in section 3.4.6 (*eds.*).

case the phases γ_n are unknown in principle,[35] so that (26) then naturally goes over to (27) [46].

It should be noted further that the formula (26) goes over to the expression (27) if the perturbation function proceeds totally irregularly as a function of time. That is for instance the case when the perturbation is produced by 'white light'.[a] Then, on average, the surplus terms in (26) drop out and one obtains (27). In this way it is easy to derive the Einstein coefficient B_{nm} for the probability per unit radiation of the quantum jumps induced by light absorption (Dirac [37], Born [30]). But, in general, according to (26) the knowledge of the probabilities $|c_n(0)|^2$ is by no means sufficient to calculate the course of the perturbation, rather one has to know also the phases γ_n.

This circumstance recalls vividly the behaviour of light in interference phenomena. The intensity of illumination on a screen is by no means always equal to the sum of the light intensities of the individual beams of rays that impinge on the screen, or, as one can well say, it is by no means equal to the sum of the light quanta that move in the individual beams; instead it depends essentially on the phases of the waves. Thus at this point an analogy between the quantum mechanics of corpuscles and the wave theory of light becomes apparent.

As a matter of fact this connection was found by de Broglie in a quite different way. It is not our purpose to discuss this. It is enough to formulate the result of de Broglie's considerations, and their further development by Schrödinger, and to put it in relation to quantum mechanics.

The dual nature of light — waves, light quanta — corresponds to the analogous dual nature of material particles; these also behave in a certain respect like waves. Schrödinger has set up the laws of propagation of these waves [24] and has arrived at equation [(17)],[36] here derived in a different way. His view, however, that these waves exhaust the essence of matter and that particles are nothing but wave packets, not only stands in contradiction with the principles of Bohr's empirically very well-founded theory, but also leads to impossible conclusions; here therefore it shall be left to one side. Instead we attribute a dual nature to matter also: its description requires both corpuscles (discontinuities) and waves (continuous processes). From the viewpoint of the statistical approach to quantum mechanics it is now clear why these can be reconciled: the waves are probability waves. Indeed, it is not the probabilities themselves,

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[a] Compare also Born's discussion in Born (1926c [34]) (*eds.*).

rather certain ‘probability amplitudes’ that propagate continuously and obey differential or integral equations, as in classical continuum physics; but additionally there are discontinuities, corpuscles whose frequency is governed by the square of these amplitudes.

The most definite support for this conception is given by collision phenomena for material particles (Born [30]). Already Einstein [16], when he deduced from de Broglie’s daring theory the possibility of ‘diffraction’ of material particles,[a] tacitly assumed that it is the particle number that is determined by the intensity of the waves. The same occurs in the interpretation given by Elsasser [17] of the experiments by Davisson and Kunsman [18,19] on the reflection of electrons by crystals; also here one assumes directly that the *number* of electrons is a maximum in the diffraction maxima. The same holds for Dymond’s [20] experiments on the diffraction of electrons by helium atoms.

The application of wave mechanics to the calculation of collision processes takes a form quite analogous to the theory of diffraction of light by small particles. One has to find the solution to Schrödinger’s wave equation (17) that goes over at infinity to a given incident plane wave; this solution behaves everywhere at infinity like an outgoing[37] spherical wave. The intensity of this spherical wave in any direction compared to the intensity of the incoming wave determines the relative number of particles deflected in this direction from a parallel ray. As a measure of the intensity one has to take a ‘current vector’[38] which can be constructed from the solution $\varphi(q, W)$, and which is formed quite analogously to the Poynting vector of the electromagnetic theory of light, and which measures the number of particles crossing a unit surface in unit time.

In this way Wentzel [31] and Oppenheimer [32] have derived wave mechanically the famous Rutherford law for the scattering of α-particles by heavy nuclei.[b]

If one wishes to calculate the probabilities of excitation and ionisation of atoms [30], then one must introduce the coordinates of the atomic electrons as variables on an equal footing with those of the colliding electron. The waves then propagate no longer in three-dimensional space but in multi-dimensional configuration space. From this one sees that the quantum mechanical waves are indeed something quite different from the light waves of the classical theory.

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[a] Note that the first prediction of such diffraction appears in fact to have been made by de Broglie in 1923; cf. section 2.2.1 (*eds.*).

[b] Cf. Born (1969, Appendix XX). The current vector, as defined there, is the usual $\mathbf{j} = \frac{h}{2\pi i}\frac{1}{2m}(\psi^*\nabla\psi - \psi\nabla\psi)$ (*eds.*).

If one constructs the current vector just defined for a solution of the generalised Schrödinger equation (22), which describes time evolution, one sees that the time derivative of the integral

$$\int |\varphi|^2 dq' \ ,$$

ranging over an arbitrary domain of the independent numerical variables q', can be transformed into the surface integral of the current vector over the boundary of that domain. From this it emerges that

$$|\varphi|^2$$

has to be interpreted as particle density or, better, as probability density. The solution φ itself is called 'probability amplitude'.

The amplitude $\varphi(q', W')$ belonging to a stationary state thus yields via $|\varphi(q', W')|^2$ the probability that for given energy W' the coordinate q' is in some given element dq'.[39] But this can be generalised immediately. In fact, $\varphi(q', W')$ is the projection of the principal axis W' of the operator H onto the principal axis q' of the operator q. One can therefore say in general (Jordan [39]): if two physical quantities are given by the operators q and Q and if one knows the principal axes of the former, for instance, according to magnitude and direction,[40] then from the equation

$$Q\varphi(q', Q') = Q'\varphi(q', Q')$$

one can determine the principal axes Q' of Q[41] and their projections $\varphi(q', Q')$ on the axes of q. Then $|\varphi(q', Q')|^2 dq'$ is the probability that for given Q' the value of q' falls in a given interval dq'.

If conversely one imagines the principal axes of Q as given, then those of q are obtained[42] through the inverse rotation; from this one easily recognises that $\overline{\varphi(Q', q')}$ is the corresponding amplitude,[43] so that $|\varphi(Q', q')|^2 dQ'$ means the probability, given q', to find the value of Q' in dQ'. If for instance one takes for Q the operator $p = \frac{h}{2\pi i}\frac{\partial}{\partial q}$, then one has the equation

$$\frac{h}{2\pi i}\frac{\partial \varphi}{\partial q'} = p'\varphi \ ,$$

thus

$$\varphi = Ce^{\frac{2\pi i}{h} q' p'} \ . \tag{28}$$

This is therefore the probability amplitude for a pair of conjugate quantities. For the probability density one obtains $|\varphi|^2 = C$, that is, for given q' every value p' is equally probable.

This is an important result, since it allows one to retain the concept of 'conjugate quantity' even in the case where the differential definition fails, namely when the quantity q has only a discrete spectrum or even when it is only capable of taking finitely many values. The latter for instance is the case for angles with quantised direction [richtungsgequantelte Winkel],[a] say for the magnetic electron, or in the Stern-Gerlach experiments. One can then, as Jordan does, call by definition a quantity p conjugate to q, if the corresponding probability amplitude has the expression (28).

As the amplitudes are the elements of the rotation matrix of one orthogonal system into another, they are composed according to the matrix rule:

$$\varphi(q',Q') = \int \psi(q',\beta')\chi(\beta',Q')d\beta' \ ; \tag{29}$$

in the case of discrete spectra, instead of the integral one has finite or infinite sums. This is the general formulation of the theorem of the interference of probabilities. As an application, let us look again at formula (24). Here $c_n(t)$ was the amplitude for the probability that the system at time t has energy W_n; $\varphi_n^0(q')$ is the amplitude for the probability that for given energy W_n the coordinate q' has a given value. Thus

$$\varphi(q',t) = \sum_n c_n(t)\varphi_n^0(q')$$

expresses the amplitude for the probability that q' at time t has a given value.

Alongside the concept of the relative state probability $|\varphi(q',Q')|^2$, there also occurs the concept of the transition probability,[44] namely, every time one considers a system as depending on an external parameter, be it time or any property of a weakly coupled external system. Then the system of principal axes of any quantity becomes dependent on this parameter; it experiences a rotation, represented by an orthogonal transformation $S(q',q'')$, in which the parameter enters (as in formula (25)). The quantities $|S(q',q'')|^2$ are the 'transition probabilities';[45] in general, however, they are not independent, instead the 'transition amplitudes' are composed according to the interference rule.

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[a] This was a standard term referring to the fact that in the presence of an external magnetic field, the projection of the angular momentum in the direction of the field has to be quantised (quantum number m). Therefore, the direction of the angular momentum with respect to the magnetic field can be said to be quantised. Cf. Born (1969, p. 121) (*eds.*).

III. — FORMULATION OF THE PRINCIPLES AND DELIMITATION OF THEIR SCOPE

After the general concepts of the theory have been developed through analysis of empirical findings, the dual task arises, first of giving a system of principles as simple as possible and connected directly to the observations, from which the entire theory can be deduced as from a mathematical system of axioms, and second of critically scrutinising experience to assure oneself that no observation conceivable by today's means stands in contradiction to the principles.

Jordan [39] has formulated such a 'system of axioms', which takes the following statements as fundamental:[46]

1) One requires for each pair of quantum mechanical quantities q, Q the existence of a probability amplitude $\varphi(q', Q')$, such that $|\varphi|^2$ gives the probability[47] that for given Q' the value of q' falls in a given infinitesimal interval.

2) Upon permutation of q and Q, the corresponding amplitude should be $\overline{\varphi(Q', q')}$.

3) The theorem (29) of the composition of probability amplitudes.

4) To each quantity q there should belong a canonically conjugate one p, defined by the amplitude (28). This is the only place where the quantum constant h appears.[48]

Finally one also takes as obvious that, if the quantities q and Q are identical, the amplitude $\varphi(q', q'')$ becomes equal to the 'unit matrix' $\delta(q' - q'')$, that is, always to zero, except when $q' = q''$. This assumption and the multiplication theorem 3) together characterise the amplitudes thus defined as the coefficients of an orthogonal transformation; one obtains the orthogonality conditions simply by stating that the composition of the amplitude belonging to q, Q with that belonging to Q, q must yield the identity.

One can then reduce all given quantities, including the operators, to amplitudes by writing them as integral operators as in formula (13). The noncommutative operator multiplication is then a consequence of the axioms and loses all the strangeness attached to it in the original matrix theory.

Dirac's method [38] is completely equivalent to Jordan's formulation, except in that he does not arrange the principles in axiomatic form.[a]

This theory now indeed summarises all of quantum mechanics in a system in which the simple concept of the calculable probability [berechenbare Wahrscheinlichkeit][49] for a given event plays the main role.[50] It also has some shortcomings, however. One formal shortcoming is the occurrence of improper functions, like the Dirac δ, which one needs for the representation of the unit matrix for continuous ranges of variables. More serious is the circumstance that the amplitudes are not directly measurable quantities, rather, only the squares of their moduli; the phase factors are indeed essential for how different phenomena are connected [für den Zusammenhang der verschiedenen Erscheinungen wesentlich], but are only indirectly determinable, exactly as phases in optics are deduced indirectly by combining measurements of intensity. It is, however, a tried and proven principle, particularly in quantum mechanics, that one should introduce as far as possible only directly observable quantities as fundamental concepts of a theory. This defect[51] is related mathematically to the fact that the definition of probability in terms of the amplitudes does not express the invariance under orthogonal transformations of the Hilbert space (canonical transformations).

These gaps in the theory have been filled by von Neumann [41,42]. There is[52] an invariant definition of the eigenvalue spectrum for arbitrary operators, and of the relative probabilities, without presupposing the existence of eigenfunctions or indeed using improper functions. Even though this theory has not yet been elaborated in all directions, one can however say with certainty that a mathematically irreproachable grounding of quantum mechanics is possible.

Now the second question has to be answered: is this theory in accord with the totality of our experience? In particular, given that the individual process is only statistically determined, how can the usual deterministic order be preserved in the composite macroscopic phenomena?[53]

The most important step in testing the new conceptual system in this direction consists in the determination of the boundaries within which the application of the old (classical) words and concepts is allowed, such as 'position, velocity, momentum, energy of a particle (electron)' (Heisenberg [46]). It now turns out that all these quantities can be

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[a] There are nevertheless some differences between the approaches of Dirac and Jordan. Cf. Darrigol (1992, pp. 343–4) (*eds.*).

individually exactly measured and defined, as in the classical theory, but that for simultaneous measurements of canonically conjugate quantities (more generally: quantities whose operators do not commute) one cannot get below a characteristic limit of indeterminacy [Unbestimmtheit].[a] To determine this, according to Bohr [47][54] one can start quite generally from the empirically given dualism between waves and corpuscles. One has essentially the same phenomenon already in every diffraction of light by a slit. If a wave impinges perpendicularly on an (infinitely long) slit of width q_1, then the light distribution as a function of the deviation angle φ is given according to Kirchhoff by the square of the modulus of the quantity

$$a\int_{-\frac{q_1}{2}}^{+\frac{q_1}{2}} e^{\frac{2\pi i}{\lambda}\sin\varphi q}dq = 2a\frac{\sin\left(\frac{\pi q_1}{\lambda}\sin\varphi\right)}{\frac{\pi q_1}{\lambda}\sin\varphi} ,$$

and thus ranges over a domain whose order of magnitude is given by[55] $\sin\varphi_1 = \frac{\lambda}{q_1}$ and gets ever larger with decreasing slit width q_1. If one considers this process from the point of view of the corpuscular theory, and if the association given by de Broglie of frequency and wavelength with energy and momentum of the light quantum is valid,

$$h\nu = W, \quad \frac{h}{\lambda} = P ,$$

then the momentum component perpendicular to the direction of the slit is

$$p = P\sin\varphi = \frac{h}{\lambda}\sin\varphi .$$

One sees thus that after the passage through the slit the light quanta have a distribution whose amplitude is given by

$$e^{\frac{2\pi i}{\lambda}\sin\varphi\cdot q} = e^{\frac{2\pi i}{h}p\cdot q} ,$$

precisely as quantum mechanics requires for two canonically conjugate variables; further, the width of the domain of the variable p that contains the greatest number of light quanta is

$$p_1 = P\sin\varphi_1 = \frac{P\lambda}{q_1} = \frac{h}{q_1} .$$

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[a] Here and in the following, the choice of translation reflects the characteristic terminology of the original. Born and Heisenberg use the terms 'Unbestimmtheit' (indeterminacy) and 'Ungenauigkeit' (imprecision), while the standard German terms today are 'Unbestimmtheit' and 'Unschärfe' (unsharpness) (*eds.*).

By general considerations of this kind one arrives at the insight that the imprecisions (average errors) of two canonically conjugate variables p and q always stand in the relation

$$p_1 q_1 \geq h \ . \tag{30}$$

The narrowing of the range of one variable, which forms the essence of a measurement, widens unfailingly the range of the other. The same follows immediately from the mathematical formalism of quantum mechanics on the basis of formula (28). The actual meaning of Planck's constant h is thus that it is the universal measure of the indeterminacy that enters the laws of nature through the dualism of waves and corpuscles.

That quantum mechanics is a mixture of strictly mechanical and statistical principles can be considered a consequence of this indeterminacy. Indeed, in the classical theory one may fix the state of a mechanical system by, for instance, measuring the initial values of p and q at a certain instant. In quantum mechanics such a measurement of the initial state is possible only with the accuracy (30). Thus the values of p and q are known also at later times only statistically.

The relation between the old and the new theory can therefore be described thus:

In classical mechanics one assumes the possibility of determining exactly the initial state; the further development is then determined by the laws themselves.

In quantum mechanics, because of the imprecision relation, the result of each measurement can be expressed by the choice of appropriate initial values for probability functions; the quantum mechanical laws determine the change (wave-like propagation) of these probability functions. The result of future experiments however remains in general indeterminate and only the expectation[56] of the result is statistically constrained. Each new experiment replaces the probability functions valid until now with new ones, which correspond to the result of the observation; it separates the physical quantities into known and unknown (more precisely and less precisely known) quantities in a way characteristic of the experiment.

That in this view certain laws, like the principles of conservation of energy and momentum, are strictly valid, follows from the fact that they are relations between commuting quantities (all quantities of the kind q or all quantities of the kind p).[a]

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[a] A similar but more explicit phrasing is used by Born (1926e, lecture 15): assuming that $H(p,q) = H(p) + H(q)$, the time derivatives not only of H but also of all components of momentum and of angular momentum have the form $f(q) + g(p)$

The transition from micro- to macromechanics results naturally from the imprecision relation because of the smallness of Planck's constant h. The fact of the propagation,[57] the 'melting away' of a 'wave or probability packet' is crucial to this. For some simple mechanical systems (free electron in a magnetic or electric field (Kennard [50]), harmonic oscillator (Schrödinger [25])), the quantum mechanical propagation of the wave packet agrees with the propagation of the system trajectories that would occur in the classical theory if the initial conditions were known only with the precision restriction (30). Here the purely classical treatment of α and β particles, for instance in the discussion of Wilson's photographs, immediately finds its justification. But in general the statistical laws of the propagation of a 'packet' for the classical and the quantum theory are different; one has particularly extreme examples of this in the cases of 'diffraction' or 'interference' of material rays, as in the already mentioned experiments of Davisson, Kunsman and Germer [18,19] on the reflection of electrons by metallic crystals.

That the totality of experience can be fitted into the system of this theory can of course be established only by calculation and discussion of all the experimentally accessible cases. Individual experimental setups,[58] in which the suspicion of a contradiction with the precision limit (30) might arise, have been discussed [46,47]; every time the reason for the impossibility of fixing exactly all determining data could be exhibited intuitively [anschaulich aufgewiesen].

There remains only to survey the most important consequences of the theory and their experimental verification.

IV. — Applications of quantum mechanics

In this section we shall briefly discuss those applications of quantum mechanics that stand in close relation to questions of principle. Here the Uhlenbeck-Goudsmit theory of the magnetic electron shall be mentioned first. Its formulation and the treatment of the anomalous Zeeman effects with the matrix calculus raise no difficulties [11]; the treatment with the method of eigenoscillations succeeds only with the help of the general Dirac-Jordan theory (Pauli [45]). Here, two three-dimensional wave functions are associated with each electron. It becomes natural thereby to look for an analogy between matter waves and polarised

with suitable functions f and g. Born states that since all q commute with one another and all p commute with one another, the expressions $f(q)+g(p)$ will vanish under the same circumstances as in classical mechanics (*eds.*).

light waves, which in fact can be carried through to a certain extent (Darwin [49], Jordan [53]). What is common to both phenomena is that the number of terms is finite, so the representative matrix is also finite (two arrangements [Einstellungen] for the electron, two directions of polarisation for light). Here the definition of the conjugate quantity by means of differentiation thus fails; one must resort to Jordan's definition by means of the probability amplitudes (formula (28)).

From among the other applications, the quantum mechanics of many-body problems shall be mentioned [28,29,40]. In a system that contains a number of similar particles [gleicher Partikel],[59] there occurs between them a kind of 'resonance' and from that results a decomposition of the system of terms into subsystems that do not combine (Heisenberg, Dirac [28,37]). Wigner has systematically investigated this phenomenon by resorting to group theoretic methods, and has set up the totality of the non-combining systems of terms [40]; Hund has managed to derive the majority of these results by comparatively elementary means [48]. A special role is played by the 'symmetric' and 'antisymmetric' subsystems of terms; in the former every eigenfunction remains unchanged under permutation of arbitrary similar particles, in the latter it changes sign under permutation of any two particles. In applying this theory to the spectra of atoms with several electrons it turns out that the Pauli equivalence rule[a] allows only the antisymmetric subsystem.[60] On the basis of this insight one can establish quantum mechanically the systematics of the line spectra and of the electron grouping throughout the whole periodic system of elements.

If one has a large number of similar particles, which are to be given a statistical treatment (gas theory), one obtains different statistics depending on whether one chooses the corresponding wave function according to the one or the other subsystem. The symmetric system is characterised by the fact that no new state arises under permutation of the particles from[61] a state described by a symmetric eigenfunction; thus all permutations that belong to the same set of quantum numbers (lie in the same 'cell') together always have the weight 1. This corresponds to the Bose-Einstein statistics [56,16]. In the antisymmetric term system two quantum numbers may never become equal, because otherwise the eigenfunction vanishes; a set of quantum numbers corresponds therefore either to no proper function at all or at most to one, thus the weight of a state is 0 or 1. This is the Fermi-Dirac statistics [57,37].

[a] That is, the Pauli principle applied to electrons that are 'equivalent' in the sense of having the same quantum numbers n and l; cf. Born (1969, p. 178). (*eds.*).

Bose-Einstein statistics holds for light quanta, as emerges from the validity of Planck's radiation formula. Fermi-Dirac statistics certainly holds for (negative) electrons, as emerges from the above-mentioned systematics of the spectra on the basis of Pauli's equivalence rule, and with great likelihood also for the positive elementary particles (protons); one can infer this from observations of band spectra [28,43] and in particular from the specific heat of hydrogen at low temperatures [55].[62] The assumption of Fermi-Dirac statistics for the positive and negative elementary particles of matter has the consequence that Bose-Einstein statistics holds for all neutral structures, e.g. molecules (symmetry of the eigenfunctions under permutation of an even number[63] of particles of matter). Within quantum mechanics, in which a many-body problem is treated in configuration space, the new statistics of Bose-Einstein and Fermi-Dirac has a perfectly legitimate place, unlike in the classical theory, where an arbitrary modification of the usual statistics is impossible; nevertheless the restrictions made on the form of the eigenfunctions appear as an arbitrary additional assumption. In particular, the example of light quanta indicates that the new statistics is related in an essential way to the wave-like properties of matter and light. If one decomposes the electromagnetic oscillations of a cavity into spatial harmonic components, each of these behaves like a harmonic oscillator as regards time evolution; it now turns out that under quantisation of this system of oscillators a solution results that behaves exactly like a system of light quanta obeying Bose-Einstein statistics [4]. Dirac has used this fact for a consistent treatment of electrodynamical problems [51,52], to which we shall return briefly.

The corpuscular structure of light thus appears here as quantisation of light waves, such as vice versa the wave nature of matter manifests itself in the 'quantisation' of the corpuscular motion. Jordan has shown [54] that one can proceed analogously with electrons; one has then to decompose the Schrödinger function of a cavity into fundamental and harmonics and to quantise each of these as a harmonic oscillator, in such a way in fact that Fermi-Dirac statistics is obtained. The new quantum numbers, which express the 'weights' in the usual many-body theory, have thus only the values 0 and 1. Therefore one has again here a case of finite matrices, which can be treated only with Jordan's general theory. The existence of electrons thus plays the same role in the formal elaboration of the theory as that of light quanta; both are discontinuities no different in kind from the stationary states of a quantised system.

However, if the material particles stand in interaction with each other, the development of this idea might run into difficulties of a deep nature.

The results of Dirac's investigations [51,52] of quantum electrodynamics consist above all in a rigorous derivation of Einstein's transition probabilities for spontaneous emission.[a] Here the electromagnetic field (resolved into quantised harmonic oscillations) and the atom are considered as a coupled system and quantum mechanics is applied in the form of the integral equation (13). The interaction energy appearing therein is obtained by carrying over classical formulas. In this connection, the nature of absorption and scattering of light by atoms is clarified. Finally, Dirac [52] has managed to derive a dispersion formula with damping term; this includes also the quantum mechanical interpretation of Wien's experiments on the decay in luminescence of canal rays.[b] His method consists in considering the process of the scattering of light by atoms as a collision of light quanta. However, since one can indeed attribute energy and momentum to the light quantum but not easily a spatial position, there is a failure of the wave mechanical collision theory (Born [30]), in which one presupposes knowledge of the interaction between the collision partners as a function of the relative position. It is thus necessary to use the momenta as independent variables, and an operator equation of matrix character instead of Schrödinger's wave equation. Here one has a case where the use of the general points of view which we have emphasised in this report cannot be avoided. At the same time, the theory of Dirac reveals anew the deep analogy between electrons and light quanta.

Conclusion

By way of summary, we wish to emphasise that while we consider the last-mentioned enquiries, which relate to a quantum mechanical treatment of the electromagnetic field, as not yet completed [unabgeschlossen], we consider *quantum mechanics* to be a closed theory [geschlossene Theorie], whose fundamental physical and mathematical assumptions are no longer susceptible of any modification. Assumptions about the physical meaning of quantum mechanical quantities that contradict Jordan's or equivalent postulates will in our opinion also contradict experience. (Such contradictions can arise for example if the square of

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[a] As opposed to the induced emission discussed on p. 424 (*eds.*).

[b] See above, p. 143 (*eds.*).

the modulus of the eigenfunction is interpreted as charge density.[a]) On the question of the 'validity of the law of causality' we have this opinion: as long as one takes into account only experiments that lie in the domain of our currently acquired physical and quantum mechanical experience, the assumption of indeterminism in principle, here taken as fundamental, agrees with experience. The further development of the theory of radiation will change nothing in this state of affairs, because the dualism between corpuscles and waves, which in quantum mechanics appears as part of a contradiction-free, closed theory [abgeschlossene Theorie], holds in quite a similar way for radiation. The relation between light quanta and electromagnetic waves must be just as statistical as that between de Broglie waves and electrons. The difficulties still standing at present in the way of a complete theory of radiation thus do *not* lie in the dualism between light quanta and waves — which is entirely intelligible — instead they appear only when one attempts to arrive at a relativistically invariant, closed formulation of the electromagnetic laws; all questions for which such a formulation is unnecessary can be treated by Dirac's method [51,52]. However, the first steps also towards overcoming these relativistic difficulties have already been made.

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[a] See Schrödinger's report, especially his section I, and section 4.4 above (*eds.*).

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Discussion of Messrs Born and Heisenberg's report[a]

Mr Dirac. — I should like to point out the exact nature of the correspondence between the matrix mechanics and the classical mechanics. In classical mechanics one can work out a problem by two methods: (1) by taking all the variables to be numbers and working out the motion, e.g. by Newton's laws, which means one is calculating the motion resulting from one particular set of numerical values for the initial coordinates and momenta, and (2) by considering the variables to be functions of the J's (action variables)[74] and using the general transformation theory of dynamics and thus determining simultaneously the motion resulting from all possible initial conditions.[75] The matrix theory corresponds to this second classical method. It gives information about all the states of the system simultaneously. A difference between the matrix method and the second classical method arises since in the latter one requires to treat simultaneously only states having nearly the same J's (one uses, for instance, the operators $\frac{\partial}{\partial J}$), while in the matrix theory one must treat simultaneously states whose J's differ by finite amounts.

To get results comparable with experiment when one uses the second classical method,[76] one must substitute numerical values for the J's in the functions of the J's obtained from the general treatment. One has to do the same in the matrix theory. This gives rise to a difficulty since the results of the general treatment are now matrix elements, each referring in general to two different sets of J's. It is only the diagonal elements, for which these two sets of J's coincide, that have a direct physical interpretation.

Mr Lorentz. — I was very surprised to see that the matrices satisfy equations of motion. In theory that is very beautiful, but to me it is a great mystery, which, I hope, will be clarified. I am told that by all these considerations one has come to construct matrices that represent what one can observe in the atom, for instance the frequencies of the emitted radiation. Nevertheless, the fact that the coordinates, the potential energy, and so on, are now represented by matrices indicates that the quantities have lost their original meaning and that one has made a huge step in the direction of abstraction.

Allow me to draw attention to another point that has struck me. Let

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[a] The two discussion contributions by Dirac follow his manuscript in AHQP, microfilm 36, section 10. Deviations in the French edition (which may or may not be due to Dirac) are reported in endnote, as well as interesting variants or cancellations in the manuscript, and punctuation has been slightly altered (*eds.*).

us consider the elements of the matrices representing the coordinates of a particle in an atom, a hydrogen atom for instance, and satisfying the equations of motion. One can then change the phase of each element of the matrices without these ceasing to satisfy the equations of motion; one can, for instance, change the time. But one can go even further and change the phases, not arbitrarily, but by multiplying each element by a factor of the form $e^{i(\delta_m - \delta_n)}$, and this is quite different from a change of time origin.[a]

Now these matrix elements ought to represent emitted radiation. If the emitted radiation were what is at the basis of all this, one could expect to be able to change all phases in an arbitrary way. The abovementioned fact then leads us very naturally to the idea that it is not the radiation that is the fundamental thing: it leads us to think that behind the emitted oscillations are hidden some true oscillations, of which the emitted oscillations are difference oscillations.

In this way then, in the end there would be oscillations of which the emitted oscillations are differences, as in Schrödinger's theory,[a] and it seems to me that this is contained in the matrices. This circumstance indicates the existence of a simpler wave substrate.

Mr Born. — Mr Lorentz is surprised that the matrices satisfy the equations of motion; with regard to this I would like to note the analogy with complex numbers. Also here we have a case where in an extension of the number system the formal laws are preserved almost completely. Matrices are some kind of hypercomplex numbers, which are distinguished from the ordinary numbers by the fact that the law of commutativity no longer holds.

Mr Dirac. — The arbitrary phases occurring in the matrix method correspond exactly[77] to the arbitrary phases in the second classical method, where the variables are functions of the J's and w's (action and angle variables). There are arbitrary[78] phases in the w's, which may have different values for each different set of values for the J's. This is completely analogous[79] to the matrix theory, in which each arbitrary phase is associated with a row and column, and therefore with a set of

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[a] This corresponds of course to the choice of a phase factor $e^{i\delta_n}$ for each stationary state. This point (among others) had been raised in the correspondence between Lorentz and Ehrenfest in the months preceding the conference. See Lorentz to Ehrenfest, 4 July 1927, AHQP-EHR-23 (in Dutch) (*eds.*).

[a] See section 4.4 (*eds.*).

values for the J's.

Mr Born. — The phases α_n which Mr Lorentz has just mentioned are associated with the different energy levels, quite like in classical mechanics. I do not think there is anything mysterious hiding behind this.

Mr Bohr. — The issue of the meaning of the arbitrary phases, raised by Mr Lorentz, is of very great importance, I think, in the discussion of the consistency of the methods of quantum theory. Although the concept of phase is indispensable in the calculations, it hardly enters the interpretation of the observations.

Notes to the translation

1 Here and in a number of places in the following, the French edition omits quotation marks present in the German typescript. They are tacitly restored in this edition.
2 The French edition gives '[47]'.
3 [diskrete] — [déterminées]
4 The French edition omits '[60]'.
5 [makroskopische] — [microscopiques]
6 [durch sinngemässe Übertragung] — [par une extension logique]
7 In the French edition, the parenthetical remark is given as a footnote.
8 The typescript does not give the reference number, only the brackets. The French edition omits the reference entirely. The mentioned results are to be found in section 4.1 of Born, Heisenberg and Jordan (1926 [4]).
9 Word omitted in the French edition.
10 [sind sinngemässe Übertragungen] — [sont des extensions logiques]
11 Misprint in the French edition: summation index 'n' in the equation.
12 [orthogonale Transformationen der φ_k] — [transformations orthogonales φ_k]
13 [beschränkte unendliche] — [partiellement infinies]
14 [Sätze] — [principes]
15 [noch zurückzukommen haben] — [n'avons pas à revenir]
16 Word omitted in the French edition.
17 The overbar is missing in the original typescript (only here), but is included in the French edition.
18 The typescript reads: 'Lanczos [ ]', the reference number is added in the French edition.
19 The typescript consistently gives this reference as '(q)', the French edition as '(9)'.
20 Equation number missing in the French edition.
21 [eine Verschiebung längs ihrer Verbindungslinie mit dem Nullpunkt] — [un déplacement de leur droite de jonction avec l'origine]
22 [sie] — [ces règles]
23 [Vorgänge] — [phénomènes]
24 Both the manuscript and the French edition read 'H_1' and 'H_2' in this paragraph and two paragraphs later, and '$H^{(1)}$' in the intervening paragraph. We have uniformised the notation.
25 The French edition consistently reads '$\overline{\delta f_n}$'.
26 The right-hand side of this equation reads '$\sum_m \{f(W_n) - f(W_m)\}\Phi_{nm}$' in both the typescript and the French edition, but it should be as shown (see above, p. 115).
27 The French edition gives '(2)'.
28 Both the typescript and the French edition read (only here) 'n_1, n_2'.
29 Both the typescript and the French edition read 'Φ_{nm}'.
30 Singular in the French edition.
31 The typescript includes the square brackets but no reference number. The French edition omits the reference entirely.
32 Only brackets in the typescript, references omitted in the French edition.
33 [abhängige] — [indépendante]
34 The French edition reads '$(c_{nk})^2$' instead of '$|c_n|^2$' and 'nk' instead of 'n'.
35 The French edition reads 'p_{nk}' instead of 'γ_n'.

36 Both the typescript and the French edition give '(11)', but this should evidently be either '(17)' or '(22a)'.

37 The adjective is omitted in the French edition.

38 ['Strahlvektor'] — ['vecteur radiant']

39 The absolute square is missing in the German typescript, but is added in the French edition.

40 [des einen, etwa, nach Grösse und Richtung] — [de l'un, par exemple en grandeur et en direction]

41 [von Q] — [et Q]

42 The French edition has a prime on 'q'.

43 The overbar is missing in the German typescript, but is added in the French edition.

44 Throughout this paragraph, the French edition translates 'Übergang' as 'transformation' instead of 'transition'.

45 'S' missing in the French edition.

46 [das folgende Sätze zugrunde legt] — [qui est à la base des théorèmes suivants]. Note that 'Satz' can indeed mean both 'statement' and 'theorem'.

47 The French edition omits absolute bars.

48 The 'h' is present in the French edition but not in the typescript.

49 In the typescript, this is typed over an (illegible) previous alternative. Jordan in his habilitation lecture (1927f [62]) uses the term 'angebbare Wahrscheinlichkeit' ('assignable probability' in Oppenheimer's translation (Jordan 1927g)).

50 [in dem der einfache Begriff der berechenbaren Wahrscheinlichkeit für ein bestimmtes Ereignis die Hauptrolle spielt] — [dans lequel la simple notion de la probabilité calculable joue le rôle principale pour un événement déterminé]

51 [Überstand] — [défaut]. The word 'Überstand' may be characterising the phases as some kind of surplus structure, but it is quite likely a mistyping of 'Übelstand', which can indeed be translated as 'defect', as in the French version.

52 [Es gibt] — [Cet auteur donne]

53 [Wie kann insbesondere bei der nur statistischen Bestimmtheit des Einzelvorgangs in den zusammengesetzten makroskopischen Erscheinungen die gewohnte deterministische Ordnung aufrecht erhalten werden?] — [En particulier comment, vu la détermination uniquement statistique des processus individuels dans les phénomènes macroscopiques compliqués, l'ordre déterministe auquel nous sommes accoutumés peut-il être conservé?]

54 This reference is to a supposedly forthcoming 'Über den begrifflichen Aufbau der Quantentheorie'. Yet, no such published or unpublished work by Bohr is extant. Some pages titled 'Zur Frage des begrifflichen Aufbaus der Quantentheorie' are contained in the folder 'Como lecture II' in the Niels Bohr archive, microfilmed in AHQP-BMSS-11, section 4. See also Bohr (1985, p. 478). We wish to thank Felicity Pors, of the Niels Bohr archive, for correspondence on this point.

55 The French edition incorrectly reads '$\sin\varphi_1 = \frac{q_1}{\lambda}$'.

56 [Erwartung] — [attente]

57 [Ausbreitung] — [extension]

58 [einzelne Versuchsanordnungen] — [des essais isolés]

59 Again, the terminology has changed both in German and in English. The term 'similar particles' for 'identical particles' is used for instance by Dirac (1927a [37]).
60 [nur das antisymmetrische Teilsystem zulässt] — [ne permet pas le système antisymétrique]
61 [aus] — [dans]
62 The French edition gives '[56]'.
63 Both the German version followed here and the French version ('a whole number of particles') seem rather infelicitous.
64 Both the typescript and the French edition add '(Magnetelektron)'. The French edition reads '*Nature*'.
65 Both the typescript and the French edition read 'p. 5'.
66 Typescript and published volume read 'Pt' and 'Mg', as well as '243'.
67 In the French edition: '409'.
68 This is indeed the (abridged) published version of Elsasser's Göttingen dissertation.
69 In the French edition: 'Das Adiabatenprinzip in den Quanten', as well as '1927' (the latter as in the typescript).
70 In both the typescript and the French edition the title of the paper is given as 'Über den anschaulichen Inhalt der Quantenmechanik'.
71 Date given as '1927' in typescript and volume.
72 In the French edition: '288'.
73 The French edition omits '614'.
74 The manuscript includes also 'and w's' and 'and angle', both cancelled.
75 The French edition breaks up and rearranges this sentence.
76 The French edition omits the temporal clause.
77 The French edition reads 'trouvent une analogie'.
78 In the manuscript this replaces the cancelled word 'unknown'.
79 In the manuscript this replaces 'corresponds exactly'.

Wave mechanics[a]

By Mr E. SCHRÖDINGER

Introduction

Under this name at present two theories are being carried on, which are indeed closely related but not identical. The first, which follows on directly from the famous doctoral thesis by L. de Broglie, concerns waves in three-dimensional space. Because of the strictly relativistic treatment that is adopted in this version from the outset, we shall refer to it as the *four-dimensional* wave mechanics. The other theory is more remote from Mr de Broglie's original ideas, insofar as it is based on a wave-like process in the space of *position coordinates* (q-space) of an arbitrary mechanical system.[1] We shall therefore call it the *multi-dimensional* wave mechanics. Of course this use of the q-space is to be seen only as a mathematical tool, as it is often applied also in the old mechanics; ultimately, in this version also, the process to be described is one in space and time. In truth, however, a complete unification of the two conceptions has not yet been achieved. Anything over and above the motion of a single electron could be treated so far only in the *multi*-dimensional version; also, this is the one that provides the mathematical solution to the problems posed by the Heisenberg-Born matrix mechanics. For these reasons I shall place

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[a] Our translation follows Schrödinger's German typescript in AHQP-RDN, document M-1354. Discrepancies between the typescript and the French edition are endnoted. Interspersed in the German text, Schrödinger provided his own summary of the paper (in French). We translate this in the footnotes. The French version of this report is also reprinted in Schrödinger (1984, vol. 3, pp. 302–23) (*eds.*).

it first, hoping in this way also to illustrate better the characteristic difficulties of the as such more beautiful four-dimensional version.[a]

I. — MULTI-DIMENSIONAL THEORY

Given a system whose configuration is described by the generalised position coordinates $q_1, q_2, \ldots, q_n$, classical mechanics considers its task as being that of determining the q_k as functions of time, that is, of exhibiting *all* systems of functions $q_1(t), q_2(t), \ldots, q_n(t)$ that correspond to a dynamically possible motion of the system. Instead, according to wave mechanics the solution to the problem of motion is not given by a system of n functions of the single variable t, but by a *single* function ψ of the n variables $q_1, q_2, \ldots, q_n$ and perhaps of time (see below). This is determined by a *partial* differential equation with $q_1, q_2, \ldots, q_n$ (and perhaps t) as *independent* variables. This change of role of the q_k, which from dependent become independent variables, appears to be the crucial point. More later on the meaning of the function ψ, which is still controversial. We first describe how it is determined, thus what corresponds to the equations of motion of the old mechanics.

First let the system be a *conservative* one. We start from its Hamiltonian function

$$H = T + V \ ,$$

that is, from the total energy expressed as a function of the q_k and the canonically-conjugate momenta p_k. We take H to be a *homogeneous* quadratic function of the q_k and of unity and replace in it each p_k by $\frac{h}{2\pi}\frac{\partial\psi}{\partial q_k}$ and unity by ψ. We call the function of the q_k, $\frac{\partial\psi}{\partial q_k}$ and ψ thus obtained L (because in wave mechanics it plays the role of a Lagrange function). Thus

$$L = T\left(q_k, \frac{h}{2\pi}\frac{\partial\psi}{\partial q_k}\right) + V\psi^2 \ . \tag{1}$$

Now we determine $\psi(q_1, q_2, \ldots, q_n)$ by the requirement that under variation of ψ,

$$\delta \int L d\tau = 0 \quad \text{with} \quad \int \psi^2 d\tau = 1 \ . \tag{2}$$

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[a] Summary of the introduction: Currently there are in fact *two* [theories of] wave mechanics, very closely related to each other but not identical, that is, the relativistic or *four-dimensional* theory, which concerns waves in ordinary space, and the *multi-dimensional* theory, which originally concerns waves in the configuration space of an arbitrary system. The former, until now, is able to deal only with the case of a single electron, while the latter, which provides the solution to the matrix problems of Heisenberg-Born, comes up against the difficulty of being put in relativistic form. We start with the latter.

The integration is to be performed over the whole of q-space (on whose perhaps infinitely distant boundary, $\partial\psi$ must disappear). However, $d\tau$ is not simply the product of the dq_k, rather the 'rationally measured' volume element in q-space:

$$d\tau = dq_1 dq_2 \ldots dq_n \left| \pm \frac{\partial^2 T}{\partial p_1 \ldots \partial p_k} \right|^{-\frac{1}{2}} \tag{3}$$

(it is the volume element of a *Riemann*ian q-space, whose *metric*, as for instance also in Hertz's mechanics,[a] is determined by the kinetic energy). — Performing the variation, taking the normalisation constraint with the multiplier [Factor] $-E$, yields the Euler equation

$$\Delta\psi + \frac{8\pi^2}{h^2}(E - V)\psi = 0 \tag{4}$$

(Δ stands for the analogue of the Laplace operator in the generalised Riemannian sense). As is well known,

$$\int L d\tau = E$$

for a function that satisfies the Euler equation (4) and the constraint in (2).

Now, it turns out that equation (4) in general does not have, for every E-value, a solution ψ that is single-valued and always finite and continuous together with its first and second derivatives; instead, in all special cases examined so far, this is the case precisely for the E-values that Bohr's theory would describe as stationary energy levels of the system (in the case of discrepancies, the recalculated values explain the facts of experience *better* than the old ones). The word 'stationary' used by Bohr is thus given a very pregnant meaning by the variation problem (4).

We shall refer to these values as eigenvalues, E_k, and to the corresponding solutions ψ_k as eigenfunctions.[b] We shall number the eigenvalues always in increasing order and shall number repeatedly those with multiple eigensolutions. The ψ_k form a normalised complete orthogonal system in the q-space, with respect to which every well-behaved function of the q_k can be expanded in a series. Of course this does not mean that every well-behaved function solves the homogeneous equation (4) and

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[a] For Schrödinger's interest in Hertz's work on mechanics, see Mehra and Rechenberg (1987, pp. 522–32) (*eds.*).

[b] As a rule, in certain domains of the energy axis[2] the eigenvalue spectrum is continuous, so that the index k is replaced by a continuous parameter. In the notation we shall generally not take this into account.

thus the variation problem, because (4) is indeed an equation *system*, each single eigensolution ψ_k satisfying a different element of the system, namely the one with $E = E_k$.[a]

One can take the view that one should be content in principle with what has been said so far and its very diverse special applications. The single stationary states of Bohr's theory would then in a way be described by the eigenfunctions ψ_k, which *do not contain time at all*.[a] One would find that one can derive much more from them that is worth knowing, in particular, one can form from them, by fixed general rules, quantities that can be aptly taken to be *jump probabilities* between the single stationary states. Indeed, it can be shown for instance that the integral

$$\int q_i \psi_k \psi_{k'} d\tau \ , \tag{5}$$

extended to the whole of q-space, yields precisely the matrix element bearing the indices k and k' of the 'matrix q' in the Heisenberg-Born theory; similarly, the elements of all matrices occurring there can be calculated from the wave mechanical eigenfunctions.

The theory as it stands, restricted to conservative systems, could treat already even the *interaction* between two or more systems, by considering these as one single system, with the addition of a suitable term in the potential energy depending on the coordinates of all subsystems. Even the interaction of a material system with the radiation field is not out of reach, if one imagines the system together with certain ether oscillators (eigenoscillations of a cavity) as a single conservative system, positing suitable interaction terms.

On this view the *time variable* would play absolutely no role in an isolated system — a possibility to which N. Campbell (*Phil. Mag.*, [**1**] (1926), [1106]) has recently pointed. Limiting our attention to an isolated system, we would not perceive the passage of time in it any more than

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[a] Summary of the above: Wave mechanics demands that events in a mechanical system that is in motion be described not by giving n generalised coordinates $q_1, q_2 \ldots q_n$ as functions of the time t, but by giving a *single* function $[\psi]$ of the n variables $q_1, q_2 \ldots q_n$ and maybe of the time t. The system of equations of motion of classical mechanics corresponds in wave mechanics to a *single* partial differential equation, eq. (4), which can be obtained by a certain variational procedure. E is a Lagrange multiplier, V is the potential energy, a function of the coordinates; h is Planck's constant, Δ denotes the Laplacian in q-space, generalised in the sense of Riemann. One finds in specific cases that finite and continuous solutions, 'eigenfunctions' ψ_k of eq. (4), exist only for certain 'eigenvalues' E_k of E. The set of these functions forms a complete orthogonal system in the coordinate space. The eigenvalues are precisely the 'stationary energy levels' of Bohr's theory.

[a] Cf. section 8.1 (*eds.*).

we can notice its possible progress in space, an assimilation of time to the spatial coordinates that is very much in the spirit of relativity. What we would notice would be merely a sequence of discontinuous transitions, so to speak a cinematic image, but without the possibility of comparing the time intervals between the transitions. Only secondarily, and in fact with increasing precision the more extended the system, would a *statistical* definition of time result from *counting* the transitions taking place (Campbell's 'radioactive clock'). Of course then one cannot understand the jump probability in the usual way as the probability of a transition calculated relative to unit time. Rather, a *single* jump probability is then utterly meaningless; only with *two* possibilities for jumps, the probability that the one may happen *before* the other is equal to *its* jump probability divided by the sum of the two.

I consider this view the only one that would make it possible to hold on to 'quantum jumps' in a coherent way. Either all changes in nature are discontinuous or not a single one. The first view may have many attractions; for the time being however, it still poses great difficulties. If one does not wish to be so radical and give up in principle the use of the *time variable* also for the single atomistic system, then it is very natural to assume that it is contained hidden also in equation (4). One will conjecture that equation system (4) is the *amplitude* equation of an *oscillation* equation, from which time has been eliminated by setting[a]

$$\psi \sim e^{2\pi i\nu t} \ . \tag{6}$$

E must then be proportional to a power of ν, and it is natural to set $E = h\nu$. Then the following is the oscillation equation that leads to (4) with the ansatz (6):[b]

$$\Delta\psi - \frac{8\pi^2}{h^2}V\psi - \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0 \ . \tag{7}$$

Now *this* is satisfied not just by a single[3]

$$\psi_k e^{2\pi i\nu_k t} \quad (\nu_k = \frac{E_k}{h}) \ ,$$

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[a] Schrödinger introduces the time-dependent equation in his fourth paper on quatisation (1926g). There (p. 112), Schrödinger leaves the sign of time undetermined, settling on the same convention as in (6) — the opposite of today's convention — on pp. 114-15. As late as Schrödinger (1926h, p. 1065), one reads that 'the most general solution of the wave-problem will be (the real part of) [eq. (27) of that paper]'. Instead the wave function is characterised as 'essentially complex' in Schrödinger (1927c, fn. 3 on p. 957) (*eds.*).

[b] Recall that Schrödinger does not in fact set $m = 1$, but absorbs the mass in the definition of Δ (*eds.*).

but by an arbitrary linear combination

$$\psi = \sum_{k=1}^{\infty} c_k \psi_k e^{2\pi i \nu_k t} \tag{8}$$

with arbitrary (even complex) constants c_k . If one considers *this* ψ as the description[4] of a certain sequence of phenomena in the system, then this is now given by a (complex) function of the $q_1, q_2, \ldots, q_n$ *and* of time, a function which can even be given arbitrarily at $t = 0$ (because of the completeness[5] and orthogonality of the ψ_k); the oscillation equation (7), or its solution (8) with suitably chosen c_k, then governs the temporal development. Bohr's stationary states correspond to the eigenoscillations of the structure (*one* $c_k = 1$, all others $= 0$).

There now seems to be no obstacle to assuming that equation (7) is valid immediately also for non-conservative systems (that is, V may contain time explicitly). Then, however,[6] the solution no longer has the simple form (8). A particularly interesting application hereof is the perturbation of an atomic system by an electric alternating field. This leads to a theory of *dispersion*, but we must forgo here a more detailed description of the same. — From (7) there *always* follows

$$\frac{d}{dt} \int d\tau \psi \psi^* = 0 \ . \tag{9}$$

(An asterisk shall always denote the complex conjugate.[7]) Instead of the earlier normalisation condition (2), one can thus require

$$\int d\tau \psi \psi^* = 1 \ , \tag{10}$$

which in the conservative case, equation (8), means

$$\sum_{k=1}^{\infty} c_k c_k^* = 1 \ .^{\mathrm{a}} \tag{11}$$

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[a] Summary of the above: Even limiting oneself to what has been said up to now, it would be possible to derive much of interest from these results, for instance the transition probabilities, formula (5) yielding precisely the matrix element $q_i(k, k')$ for the same mechanical problem formulated according to the Heisenberg-Born theory. Although we have restricted ourselves so far to conservative systems, it would be possible to treat in this way also the mutual action between several systems and even between a material system and the radiation field; one would only have to add all relevant systems to the system under consideration. *Time* does not appear at all in our considerations and one could imagine that the only events that occur are sudden transitions from one quantum state of the total system to another quantum state, as Mr N. Campbell has recently thought. Time would be defined only statistically by counting the quantum jumps (Mr Campbell's

What does the ψ-function mean now, that is, *how does the system described by it really look like in three dimensions*? Many physicists today are of the opinion that it does not describe[8] the occurrences in an individual system,[9] but only the processes in an ensemble of very many like constituted systems that do not sensibly influence one another[10] and are all under the very same conditions. I shall skip this point of view, since others are presenting it.[a] I myself have so far found useful the following perhaps somewhat naive but quite concrete idea [dafür recht greifbare Vorstellung]. The classical system of material points does not really exist, instead there exists something that continuously fills the entire space and of which one would obtain a 'snapshot' if one dragged the classical system, with the camera shutter open, through *all* its configurations, the representative point in q-space spending in each volume element $d\tau$ a time that is proportional to the *instantaneous* value of $\psi\psi^*$. (The value of $\psi\psi^*$ for only *one* value of the argument t is thus in question.) Otherwise stated: the real system is a superposition of the classical one in all its possible states, using $\psi\psi^*$ as 'weight function'.

The systems to which the theory is applied consist classically of several[12] charged point masses. In the interpretation just discussed[13] the charge of every single one of these is distributed continuously across space, the individual point mass with charge e yielding to the charge in the three-dimensional volume element $dx\,dy\,dz$ the contribution[14]

$$e \int' \psi\psi^* d\tau \ . \tag{12}$$

The prime on the integration sign means: one has to integrate only over the part of the q-space corresponding to a position of the distinguished point mass within $dxdydz$. — Since $\psi\psi^*$ in general depends on time, these charges fluctuate; only in the special case of a conservative system

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'radioactive clock'). — Another, less radical, point of view is to assume that time is hidden already in the family of equations (4) parametrised by E, this family being the *amplitude* equation of an *oscillation* equation, from which time has been eliminated by the ansatz (6). Assuming $h\nu = E$ one arrives at eq. (7), which, because it no longer contains the frequency ν, is solved by the *series* (8), where the c_k are arbitrary, generally complex, constants. Now ψ is a function of the $q_1, q_2 \ldots q_n$ *as well as* of time t and, by a suitable choice of the c_k, it can be adjusted to an arbitrary initial state. Nothing prevents us now from making the time appear also in the function V — this is the theory of non-conservative systems, one of whose most important applications is the theory of dispersion. — The important relation (9), which follows from eq. (7), allows one in all cases to normalise ψ according to eq. (10).

[a] See the report by Messrs Born and Heisenberg.[11]

oscillating with a single eigenoscillation are they distributed permanently, so to speak statically.

It must now be emphasised that by the claim that there *are*[15] these charge densities (and the current densities arising from their fluctuation), we can mean at best *half* of what classical electrodynamics would mean by that. Classically, charge and current densities are (1) *application points*, (2) *source points* of the electromagnetic field. As *application points* they are completely out of the question here; the assumption that these charges and currents act, say, according to Coulomb's or Biot-Savart's law directly on one another, or are directly affected in such a way by external fields, this assumption is either superfluous or wrong (N.B. *de facto* wrong), because the changes in the ψ function and thereby in the charges are indeed to be determined through the oscillation equation (7) — thus we must not think of them as determined also in another way, by forces acting on them. An external electric field is to be taken account of in (7) in the potential function V, an external magnetic field in a similar way to be discussed below, — this is the way their application to the charge distribution is expressed in the present theory.

Instead, our spatially distributed charges prove themselves excellently as *source points* of the field, at least for the external action of the system, in particular with respect to its *radiation.* Considered as source points in the sense of the usual electrodynamics, they yield largely[16] correct information about its frequency, intensity and polarisation.[a] In most cases, the charge is in practice confined to a region that is small compared to the wavelengths of the emitted light. The radiation is then determined by the resulting *dipole moment* [*elektrisches Moment*] of the charge distribution. According to the principles determined above, this is calculated from the classical dipole moment of an arbitrary configuration by performing an average using $\psi\psi^*$

$$M_{\mathrm{qu}} = \int M_{\mathrm{cl}} \psi\psi^* d\tau \ . \tag{13}$$

A glance at (8) shows that in M_{qu} the *differences* of the ν_k will appear as emission frequencies; since the ν_k are the spectroscopic term values, our picture provides an *understanding*[17] *of Bohr's frequency condition.* The integrals that appear as amplitudes of the different partial oscillations of the dipole moment represent according to the remarks on (5) the elements of Born and Heisenberg's 'dipole moment matrix'. By evalua-

[a] See the discussion after the report, as well as section 4.4 (*eds.*).

ting these integrals one obtained the correct polarisations and intensities of the emitted light in many special cases, in particular intensity zero in all cases where a line allowed by the frequency condition is missing according to experience (*understanding*[18] *of the selection principle*). — Even though all these results, if one so wishes, can be detached from the picture of the fluctuating charges and be represented in a more abstract form, yet they put quite beyond doubt that the picture is tremendously useful for one who has the need for Anschaulichkeit![19,a]

In no way should one claim that the provisional attempt of a classical-electrodynamic coupling of the field to the charges generating the field is already the last word on this issue. There are internal[20] reasons for doubting this. First, there is a serious difficulty in the question of the *reaction* of the emitted radiation on the emitting system, which is not yet expressed by the wave equation (7), according to which also such wave forms of the system that continuously emit radiation could and would in fact always persist unabated. Further, one should consider the following. We always observe the radiation emitted by an atom only through its action on another atom or molecule. Now, from the wave mechanical standpoint we can consider two charged point masses that belong to the *same* atom, neither as acting directly on each other in their pointlike form (standpoint of classical mechanics), nor are we allowed to think this of their 'smeared out' wave mechanical charge distributions (the wrong move taunted above). Rather, we have to take account of their classical potential energy, considered as a function in q-space, in the coefficient V of the wave equation (7). But then, when we have two *different* atoms, it will surely not be correct in principle to insert the fields generated by the *spread-out* charges of the first at the position of the second in the

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[a] Summary of the above: The physical meaning of the function ψ appears to be that the system of charged point particles imagined by classical mechanics does not in fact exist, but that there is a continuous distribution of electric charge, whose density can be calculated at each point of ordinary space using ψ or rather $\psi\psi^*$, the square of the absolute value of ψ. According to this idea, the quantum (or: real) system is a superposition of all the possible configurations of the classical system, the real function $\psi\psi^*$ in q-space occurring as 'weighting function'. Since $\psi\psi^*$ in general contains time, fluctuations of charge must occur. What we mean by the *existence* of these continuous and fluctuating charges is not at all that they should act on each other according to Coulomb's or Biot-Savart's law — the *motion* of these charges is already completely governed by eq. (7). But what we mean is that they are the *sources* of the electric fields and magnetic fields proceeding from the atom, above all the sources of the observed radiation. In many a case one has obtained wonderful agreement with experiment by calculating the radiation of these fluctuating charges using classical electrodynamics. In particular, they yield a complete and general explanation of Bohr's 'frequency condition' and of the spectral 'polarisation and selection rules'.

wave equation for the latter. And yet we do this when we calculate the radiation of an atom in the way described above and now treat wave mechanically the behaviour of another atom in this radiation field. I say this way of calculating the interaction between the charges of different atoms can be at most approximate, but not correct in principle. For *within one* system it is certainly wrong. But if we bring the two atoms closer together, then the distinction between the charges of one and those of the other gradually disappears, it is actually never a distinction of principle.[21] — The coherent wave mechanical route would surely be to combine both the emitting and the receiving system into a single one and to describe them through a *single* wave equation with appropriate coupling terms, however large the distance between emitter and receiver may be. Then one could be completely silent about the processes in the radiation field. But what would be the correct coupling terms? Of course not the usual Coulomb potentials, as soon as the distance is equal to several wavelengths![22] (One realises from here that without important amendments the *entire* theory in reality can only be applied to very *small* systems.) Perhaps one should use the *retarded* potentials. But these are not functions in the (common) q-space, instead they are something much more complicated. Evidently we encounter here the provisional limits of the theory and must be happy to possess in the procedure depicted above an approximate treatment that appears to be very useful.[a]

II. — FOUR-DIMENSIONAL THEORY

If one applies the *multi*-dimensional version of wave mechanics to a single electron of mass m and charge e moving in a space with the electrostatic potential φ and to be described by the three rectangular coordinates

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[a] Summary of the above: However, there are reasons to believe that our fluctuating and purely classically radiating charges do not provide the last word on this question. Since we observe the radiation of an atom only by its effect on another atom or molecule (which we shall thus also treat quite naturally by the methods of wave mechanics), our procedure reduces to substituting into the wave equation of *one* system the potentials that would be produced according to the classical laws by the extended charges of another system. This way of accounting for the mutual action of the charges belonging to two different systems cannot be absolutely correct, since for the charges belonging to the same system it is not. The correct method of calculating the influence of a radiating atom on another atom would be perhaps to treat them as *one* total system according to the methods of wave mechanics. But that does not seem at all possible, since the retarded potentials, which should no doubt occur, are not simply functions of the configuration of the systems, but something much more complicated. Evidently, at present these are the limits of the method!

x, y, z, then the wave equation (7) becomes

$$\frac{1}{m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)-\frac{8\pi^2}{h^2}e\varphi\psi-\frac{4\pi i}{h}\frac{\partial\psi}{\partial t}=0\ . \tag{14}$$

(N.B. The factor $\frac{1}{m}$ derives from the fact that, given the way of determining the metric of the q-space through the kinetic energy, $x\sqrt{m}$, $y\sqrt{m}$, $z\sqrt{m}$ should be used as coordinates rather than x, y, z.[23]) It now turns out that the present equation is nothing else but the ordinary three-dimensional wave equation for de Broglie's 'phase waves' of the electron, except that the equation in the above form is shortened or truncated in a way that one can call 'neglecting the influence of relativity'.

In fact, in the electrostatic field de Broglie gives the following expression[a] for the wave velocity u of his phase waves, depending on the potential φ (i.e. on position) and on the frequency ν:[24]

$$u=c\frac{h\nu}{\sqrt{(h\nu-e\varphi)^2-h^2\nu_0^2}}\qquad\left(\nu_0=\frac{mc^2}{h}\right)\ . \tag{15}$$

If one inserts this into the ordinary three-dimensional wave equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2}=0\ ,$$

and uses (6) to eliminate the frequency ν from the equation, one has[25] $(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})$

$$\Delta\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}+\frac{4\pi ie\varphi}{hc^2}\frac{\partial\psi}{\partial t}+\frac{4\pi^2}{c^2}\left(\frac{e^2\varphi^2}{h^2}-\nu_0^2\right)\psi=0\ . \tag{16}$$

Now if one considers that in the case of 'slow electron motion' (a) the occurring frequencies are always very nearly equal to the rest frequency ν_0, so that in order of magnitude the derivative with respect to time in (16) is equal to a multiplication by $2\pi i\nu_0$, and that (b) $\frac{e\varphi}{h}$ in this case[26] is always small with respect to ν_0; and if one then sets in equation (16)

$$\psi=e^{2\pi i\nu_0 t}\tilde{\psi}\ , \tag{17}$$

and disregarding squares of small quantities, one obtains for $\tilde{\psi}$ exactly equation (14) derived from the multi-dimensional version of wave mechanics. As claimed, this is thus indeed the 'classical approximation' of the wave equation holding for de Broglie's phase waves.[b] The transformation (17) here shows us that, considered from de Broglie's point of view, the

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[a] Cf. the formula for the refractive index on p. 375 of de Broglie's report (*eds.*).

[b] That is, the nonrelativistic approximation (*eds.*).

multi-dimensional theory is committed to a so to speak truncated view of the frequency, in that it subtracts once and for all from all frequencies the rest frequency ν_0 (N.B. In calculating the charge density from $\psi\psi^*$,[27] the additional factor is of course irrelevant since it has modulus[28] 1.[29])[a]

Let us now keep to the form (16) of the wave equation. It still requires an important generalisation. In order to be truly relativistic it must be invariant with respect to Lorentz transformations. But if we perform such a transformation on our electric field, hitherto assumed to be static, then it loses this feature and a magnetic field appears by itself next to it. In this way one derives almost unavoidably the form of the wave equation in an *arbitrary* electromagnetic field. The result can be put in the following transparent form, which makes the complete equivalence [Gleichberechtigung] of time and the three spatial coordinates fully explicit :

$$\begin{aligned}\Bigg[\left(\frac{\partial}{\partial x}+\frac{2\pi ie}{hc}a_x\right)^2+\left(\frac{\partial}{\partial y}+\frac{2\pi ie}{hc}a_y\right)^2 \\ +\left(\frac{\partial}{\partial z}+\frac{2\pi ie}{hc}a_z\right)^2+\left(\frac{1}{ic}\frac{\partial}{\partial t}+\frac{2\pi ie}{hc}i\varphi\right)^2-\frac{4\pi^2\nu_0^2}{c^2}\Bigg]\psi=0\ .\end{aligned} \tag{18}$$

(N.B. a is the vector potential.[30] In evaluating the squares one has to take account of the *order* of the factors, since one is dealing with operators, and further of Maxwell's relation:

$$\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}+\frac{1}{ic}\frac{\partial(i\varphi)}{\partial t}=0\ .) \tag{19}$$

This wave equation is of very manifold interest. First, as shown by Gordon,[a] it can be derived in a way very similar to what we have seen above for the amplitude equation of conservative systems, from a variational principle, which now obtains in four dimensions, and where time plays a perfectly symmetrical role with respect to the three spatial coordinates. Further: if one adds to the *Lagrange function* of Gordon's variational principle the well-known Lagrange function of the Maxwell

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[a] Summary of the above: The three-dimensional wave equation, eq. (14), obtained by applying the multi-dimensional theory to a single electron in an electrostatic potential field φ, is none other than the nonrelativistic approximation of the wave equation that results from Mr L. [d]e Broglie's ideas for his 'phase waves'. The latter, eq. (16), is obtained by substituting into the ordinary wave equation expression (15), which Mr [d]e Broglie has derived for the phase velocity u as a function of the frequency ν and of the potential φ (that is, of the coordinates x, y, z, on which φ will depend) and by eliminating from the resulting formula the frequency ν by means of (6).

[a] W. Gordon, *Zeitschr. f. Phys.*, **40** (1926), 117.

field *in vacuo* (that is, the half-difference of the squares of the magnetic and the electric field strenghts) and varies in the spacetime integral of the new Lagrange function thus obtained not only ψ, but also the potential components φ, a_x, a_y, a_z, one obtains as the *five* Euler equations along with the wave equation (18) also the four retarded potential equations for φ, a_x, a_y, a_z.[a] (One could also say: Maxwell's *second* quadruple of equations, while the first, as is well-known, holds identically in the potentials.[32]) It contains as *charge and current density* quadratic forms in ψ and its first derivatives[33] that agree completely with the rule which we had given in the *multi*-dimensional theory for calculating the true charge distribution from the ψ-function. Second, one can further define[b] a *stress-energy-momentum tensor* of the *charges*, whose ten components are also quadratic forms of ψ and its first derivatives, and which together with the well-known Maxwell tensor obeys the laws of conservation of energy and of momentum (that is, the sum of the two tensors has a vanishing divergence).[c]

But I shall not bother you here with the rather complex mathematical development of these issues, since the view still contains a serious inconsistency. Indeed, according to it, it would be the *same* potential components φ, a_x, a_y, a_z which *on the one hand* act to modify the wave equation (18) (one could say: they act *on* the charges *as movers*[35]) and which *on the other hand* are determined in turn, via the retarded potential equations, *by* these same charges, which occur as *sources* in the latter equations. (That is: the wave equation (18) determines the ψ function, from the latter one derives the charge and current densities, which as sources determine the potential components.) — In reality, however, one operates *otherwise* in the application of the wave equation (18) to the hydrogen electron, and one *must* operate otherwise to obtain the correct result: one substitutes in the wave equation (18) the *already*

[a] E. Schrödinger, *Ann. d. Phys.*, **82** (1927), [265].[31]

[b] E. Schrödinger, *loc. cit.*[34]

[c] Summary of the above: In order to generalise equation (16) so that it may apply to an arbitrary electromagnetic field, one subjects it to a Lorentz transformation, which automatically makes a magnetic field appear. One arrives at eq. (18), in which time enters in a perfectly symmetrical way with the spatial coordinates. Gordon has shown that this equation derives from a four-dimensional variational principle. By adding to Gordon's Lagrangian the well-known Lagrangian of the free field and by varying along with ψ also the four components of the potential, one derives from a single variational principle besides eq. (18) also the laws of electromagnetism with certain homogeneous quadratic functions of ψ and its first derivatives as charge and current densities. These agree well with what was said in the previous chapter regarding the calculation of the fluctuating charges using the ψ function. — One finds a definition of the stress-energy-momentum tensor, which, added to Maxwell's tensor, satisfies the conservation laws.

given potentials of the nucleus and of possible external fields (Stark and Zeeman effect). From the solution for ψ thus obtained one derives the fluctuating charge densities discussed above, which one in fact[36] has to use for the determination from sources of the *emitted radiation*; but one must *not* add *a posteriori* to the field of the nucleus and the possible external fields also the fields produced by these charges at the position of the atom itself in equation (18) —[37] something totally wrong would result.

Clearly this is a painful lacuna. The pure field theory is not enough, it has to be supplemented by performing a kind of *individualisation* of the charge densities coming from the single point charges of the classical model, where however each single 'individual' may be spread over the whole of space, so that they overlap. In the wave equation for the single individual one would have to take into account only the fields produced by the other individuals but not its self-field. These remarks, however, are only meant to characterise the general nature of the required supplement, not to constitute a programme to be taken completely literally.[a]

We wish to present also the remarkable special result yielded by the relativistic form (18) of the wave equation for the hydrogen atom. One would at first expect and hope to find the well-known Sommerfeld formula for the fine structure of terms. Indeed one *does* obtain a fine structure and one *does* obtain Sommerfeld's formula, however the result contradicts experience, because it is exactly what one would find in the Bohr-Sommerfeld theory, if one were to posit the radial as well as the azimuthal quantum number as *half-integers* [halbzahlig], that is, half of an odd integer. — Today this result is not as disquieting as when it was first encountered.[b] In fact, it is well-known that the extension of Bohr's theory through the Uhlenbeck-Goudsmit electron spin [Elektronendrall], required by many other facts of experience, has to be supplemented in turn by the move to secondary quantum 'half'-numbers ['halbe' Nebenquantenzahlen] in order to obtain good results. How the

[a] Summary of the above: However, these last developments run into a great difficulty. From their direct application would follow the logical necessity of taking into account in the wave equation, for instance in the case of the hydrogen atom, not only the potential arising from the nucleus, but also the potentials arising from the fluctuating charges; which, apart from the enormous mathematical complications that would arise, would give completely wrong results. The field theory ('Feldtheorie') appears thus inadequate; it should be supplemented by a kind of individualisation of the electrons, despite these being extended over the whole of space.

[b] E. Schrödinger, *Ann. d. Phys.*, **79** (1926), [361], p. 372.

spin is represented in wave mechanics is still uncertain. Very promising suggestions[a] point in the direction that instead of the *scalar* ψ a *vector* should be introduced. We cannot discuss here this latest turn in the theory.[b]

III. — The many-electron problem

The attempts[a] to derive numerical results by means of approximation methods for the atom with *several* electrons, whose amplitude equation (4) or wave equation (7) cannot be solved directly, have led to the remarkable result that actually, despite the multi-dimensionality of the original equation, in this procedure one always needs to calculate only with the well-known three-dimensional eigenfunctions of hydrogen; indeed one has to calculate certain *three*-dimensional *charge distributions* that result from the hydrogen eigenfunctions according to the principles presented above, and one has to calculate according the principles of classical electrostatics the self-potentials and interaction potentials of these charge distributions; these *constants* then enter as coefficients in a system of equations that in a simple way determines *in principle* the behaviour of the many-electron atom. Herein, I think, lies a hint that with the furthering of our understanding 'in the end everything will indeed become intelligible in three dimensions again'.[38] For this reason I want to elaborate a little on what has just been said.

Let

$$\psi_k(x,y,z) \quad \text{and} \quad E_k \ ; \quad (k=1,2,3,\ldots)$$

be the normalised eigenfunctions (for simplicity assumed as *real*) and corresponding eigenvalues of the *one*-electron atom with Z-fold positive nucleus, which for brevity we shall call the hydrogen problem. They sa-

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[a] C. G. Darwin, *Nature*, **119** (1927), 282, *Proc. Roy. Soc. A*, **116** (1927), 227.

[b] Summary of the above: For the hydrogen atom the relativistic equation (18) yields a result that, although disagreeing with experience, is rather remarkable, that is: one obtains the same fine structure as the one that would result from the Bohr-Sommerfeld theory by assuming the radial and azimuthal quantum numbers to be 'integral and a half', that is, half an odd integer. The theory has evidently to be completed by taking into account what in Bohr's theory is called the spin of the electron. In wave mechanics this is perhaps expressed (C. G. Darwin) by a polarisation of the ψ waves, this quantity having to be modified from a scalar to a vector.

[a] See in particular A. Unsold, *Ann. d. Phys.*, **82** (1927), 355.

tisfy the three-dimensional amplitude equation (compare equation (4)):

$$\left\{\begin{array}{l} \frac{1}{m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\frac{8\pi^2}{h^2}\left(E+\frac{Ze^2}{r}\right)\psi \quad = \quad 0 \\ \\ \qquad\qquad (r=\sqrt{x^2+y^2+z^2})\ . \end{array}\right. \tag{20}$$

If only one eigenoscillation is present, one has the *static* charge distribution[39]

$$\rho_{kk} = -e\psi_k^2\ . \tag{21}$$

If one imagines *two* being excited with maximal strength, one adds to $\rho_{kk}+\rho_{ll}$ a charge distribution oscillating with frequency $|E_k - E_l|/h$, whose *amplitude* distribution is given by

$$2\rho_{lk} = -2e\psi_k\psi_l\ . \tag{22}$$

The spatial integral of ρ_{kl} vanishes when $k \neq l$ (because of the orthogonality of the ψ_k) and it is $-e$ for $k = l$. The charge distribution resulting from the presence of two eigenoscillations together has thus at every instant the sum zero. — One can now form the electrostatic potential energies

$$p_{k,l;k',l'} = \int\ldots\int dx\,dy\,dz\,dx'\,dy'\,dz'\,\frac{\rho_{kl}(x,y,z)\rho_{k'l'}(x',y',z')}{r'}\ , \tag{23}$$

where $r' = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$ and the indices k, l, k', l' may exhibit arbitrary degeneracies (to be sure, in the case $k = k', l = l'$, p is *twice* the potential self-energy of the charge distribution ρ_{kl}; but that is of no importance). *It is the constants p that control also the many-electron atom.*

Let us sketch this. Let the classical model now consist of n electrons and a Z-fold positively charged nucleus at the origin. We shall use the wave equation in the form (7). It becomes $3n$-dimensional,[40] say thus

$$\frac{1}{m}(\Delta_1+\Delta_2+\ldots+\Delta_n)\psi - \frac{8\pi^2}{h^2}(V_n+V_e)\psi - \frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = 0\ . \tag{24}$$

Here

$$\Delta_\sigma = \frac{\partial^2}{\partial x_\sigma^2}+\frac{\partial^2}{\partial y_\sigma^2}+\frac{\partial^2}{\partial z_\sigma^2}\ ; \quad \sigma = 1,2,3,\ldots,n\ . \tag{25}$$

We have considered the potential energy function as decomposed in two parts, $V_n + V_e$; V_n should correspond to the interaction of all n electrons with the nucleus, V_e to their interaction with one another, therefore[a]

$$V_n = -Ze^2 \sum_{\sigma=1}^{n} \frac{1}{r_\sigma} \ , \tag{26}$$

$$V_e = +e^2 \sum_{(\sigma,\tau)}{}' \frac{1}{r_{\sigma\tau}} \tag{27}$$

$$\left[r_\sigma = \sqrt{x_\sigma^2 + y_\sigma^2 + z_\sigma^2}, \ \ r_{\sigma\tau} = \sqrt{(x_\sigma - x_\tau)^2 + (y_\sigma - y_\tau)^2 + (z_\sigma - z_\tau)^2} \right].$$

As the starting point for an approximation procedure we choose now the eigensolutions of equation (24) with $V_e = 0$, that is with the interaction between the electrons disregarded. The eigenfunctions are then *products* of hydrogen eigenfunctions, and the eigenvalues are *sums* of the corresponding eigenvalues of hydrogen. As a matter of fact, one easily shows that[41]

$$\psi_{k_1 \ldots k_n} = \psi_{k_1}(x_1\, y_1\, z_1) \ldots \psi_{k_n}(x_n\, y_n\, z_n) e^{\frac{2\pi i t}{h}(E_{k_1} + \ldots + E_{k_n})} \tag{28}$$

always satisfies equation (24) (with $V_e = 0$). And if one takes all possible sequences of numbers [Zahlenkombinationen] for the $k_1, k_2, \ldots, k_n$, then these products of ψ_k form a complete orthogonal system in the $3n$-dimensional q-space — one has thus integrated the approximate equation completely.

One now aims to solve the full[42] equation (24) (with $V_e \neq 0$) by *expansion* with respect to this complete orthogonal system, that is one makes *this* ansatz:

$$\psi = \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} a_{k_1 \ldots k_n} \psi_{k_1 \ldots k_n} \ . \tag{29}$$

But of course the coefficients a cannot be *constants*, otherwise the above sum would again be only a solution of the truncated equation with $V_e = 0$. It turns out, however, that it is enough to consider the a as functions of *time* alone ('method of the variation of constants').[a] Substituting (29) into (24) one finds that the following conditions on the time dependence

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[a] Analogously to eq. (12), the prime on the summation sign should be interpreted as meaning that the sum is to be taken over all pairs with $\sigma \neq \tau$ (*eds.*).

[a] P. A. M. Dirac, *Proc. Roy. Soc. A*, **112** (1926), [661] p. 674.

of the a must hold:[43,44]

$$\frac{da_{k_1\ldots k_n}}{dt} = \frac{2\pi i}{h}\sum_{l_1=1}^{\infty}\cdots\sum_{l_n=1}^{\infty} v_{k_1\ldots k_n,l_1\ldots l_n} a_{l_1\ldots l_n} e^{\frac{2\pi it}{h}(E_{l_1\ldots l_n}-E_{k_1\ldots k_n})}$$

$$[k_1\ldots k_n = 1,2,3\ldots]. \quad (30)$$

Here we have set for brevity

$$E_{k_1}+\ldots+E_{k_n} = E_{k_1\ldots k_n}\ . \quad (31)$$

The v are *constants*, indeed they are *prima facie* $3n$-tuple integrals ranging over the whole of q-space (Additional explanation:[45] Where do these $3n$-tuple integrals come from? They derive from the fact that after substituting (29) into (24) one replaces the latter equation by the mathematically equivalent condition that its left-hand side shall be orthogonal to all functions of the complete orthogonal system in R_{3n}. The system (30) expresses *this* condition.) Writing this out one has

$$v_{k_1\ldots k_n,l_1\ldots l_n} = \int^{3n\text{-fold}}\!\!\cdots\!\int dx_1\ldots dz_n V_e \psi_{k_1}(x_1,y_1,z_1)\ldots\psi_{k_n}(x_n,y_n,z_n) \quad (32)$$
$$\psi_{l_1}(x_1,y_1,z_1)\ldots\psi_{l_n}(x_n,y_n,z_n)\ .$$

If one now considers the simple structure of V_e given in (27), one recognises that the v can be reduced to sextuple integrals, in fact each of them is a finite sum of some of the Coulomb potential energies defined in (23). Indeed, if in the finite sum representing V_e, we focus on an individual term, for example $e^2/r_{\sigma\tau}$, this contains only the six variables $x_\sigma,\ldots,z_\tau$. One can thus immediately perform in (32) precisely $3n-6$ integrations on this term, yielding (because of the orthogonality and normalisation of the ψ_k) the factor 1, if $k_\rho = l_\rho$ for all indices ρ that coincide neither with σ nor with τ, and yielding instead the factor 0 if even just for a single ρ different from σ and τ one has: $k_\rho \neq l_\rho$. (One sees thus that very many terms disappear.) For the non-vanishing terms, it is easy to see that they coincide with one of the p defined in (23). QED[a]

[a] Summary of the above: Calling ψ_k and E_k the eigenfunctions and eigenvalues of the problem for one electron, charge $-e$, in the field of a nucleus $+Ze$ (hydrogen problem), let us form the charge distributions (21) and (22), the former corresponding to the existence of a single normal mode, the latter to the cooperation of two of them. Taken as charge densities in ordinary electrostatics, each of these would have a certain potential energy and there would even be a certain mutual potential energy between two of them, assumed to coexist. These are the constants $p_{kl;k'l'}$ in (23). — With these givens, let us attack the problem of the n-electron atom. Dividing the potential energy in the wave equation (24)

Let us now have a somewhat closer look at the equation system (30), whose coefficients, as we have just seen, have such a relatively simple structure, and which determines the varying amplitudes of our ansatz[46] (29) as functions of time. We can allow ourselves to introduce a somewhat simpler symbolic notation, by letting the *string* of indices $k_1, k_2, \ldots, k_n$ be represented by the *single* index k, and similarly $l_1, l_2, \ldots, l_n$ by l. One then has

$$\frac{da_k}{dt} = \frac{2\pi i}{h} \sum_{l=1}^{\infty} v_{kl} a_l e^{\frac{2\pi i t}{h}(E_l - E_k)} \ . \tag{33}$$

(One must not confuse, however, E_l, E_k with the *single*[47] eigenvalues of the hydrogen problem, which were earlier denoted in the same way.[48]) This is now a system of *infinitely many* differential equations, which we cannot solve directly: so, practically nothing seems to have been gained. In turn, however, we have as yet also *neglected nothing*: with *exact* solutions a_k of (33), (29) would be an *exact* solution of (24). This is precisely where I want to place the main emphasis, greater than on the practical implementation of the approximation procedure, which shall be sketched below only for the sake of completeness. *In principle* the equations (33) determine the solution of the many-electron problem exactly;[49] — and they no longer contain anything multi-dimensional; their coefficients are simple Coulomb energies of charge distributions that already occur in the hydrogen problem. Further, the equations (33) determine the solutions of the many-electron problem according to (29) as a combination of products of the hydrogen eigenfunctions. While these products (denoted above by $\psi_{k_1 k_2 \ldots k_n}$)[50] are still functions on the $3n$-dimensional q-space, any two of them yield in the calculation of the three-dimensional charge distributions in the many-electron atom, as is easily seen, a charge distribution which if it is not identically zero corresponds again to a hydrogen distribution (denoted above by ρ_{kk} or ρ_{kl}).

These considerations are the analogue of the construction of the higher atoms from hydrogen trajectories in Bohr's theory. They reinforce the

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for this problem into two terms and neglecting at first the term V_e, due to the mutual action between the electrons, the eigensolutions would be given by (28), that is, by the products of n hydrogen functions. From these products, taken in all combinations, form the series (29), which will yield the *exact* solution of equation (24), provided that the coefficients $a_{k_1 k_2 \ldots k_n}$ are functions of time satisfying the equations (30); (see the abbreviation (31)). The coefficients v in (30) are *constants*, defined originally by the $3n$-tuple integrals (32), which however, thanks to the simple form of V_e (see (27)), reduce to sextuple integrals, namely precisely to the constants $p_{k,l;k',l'}$ (see (23)).

hope that by delving more deeply one will be able to interpret and *understand* the results of the multi-dimensional theory in three dimensions.[a]

Now, as far as the approximation method is concerned, it consists in fact of considering the contribution V_e made to the potential energy function V by the interaction of the electrons with one another, to be as far as possible *small* as compared to the action of the nucleus. The v_{kl} are then considered small compared to the eigenvalue differences $E_l - E_k$, except if $E_l = E_k$. The a_l will then vary *slowly* by comparison to the powers of e appearing on the right-hand side of equation (33), *as long as the latter are not equal to* 1, and all those terms on the right-hand side for which this is not the case will yield only small fluctuations of short period of the a_k and can be neglected in the approximation.[51] Thereby, first, the sums on the right become *finite*, because in fact always only a finite number of eigenvalues coincide. Second, the infinitely many equations separate into groups; each group contains only a finite number of a_l and can be integrated very easily.[52] This is the first step of the approximation procedure, which in theory can be continued indefinitely, but becomes more and more cumbersome. We shall not enter into details.

One can also transform the untruncated system of differential equations (33) at a *single* stroke into a system of ordinary linear equations (with infinitely many unknowns!) by setting

$$a_l = c_l e^{\frac{2\pi i t}{h}(E - E_l)} \ , \tag{34}$$

where the quantity E and the quantities c_l are unknown *constants*. Substituting into (33) one finds

$$(E - E_k)c_k = \sum_{l=1}^{\infty} v_{kl} c_l \ ; \quad (k = 1, 2, 3, \ldots) \ . \tag{35}$$

This equation system coincides with the Heisenberg-Born 'principal axes problem'. If the v_{kl} are very small quantities, then, if not *all* c_l are to be very small, E must be close to *one* of the E_l, let us say to E_k. In the first approximation then only c_k, and all those c_l for which

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[a] Summary of the above: Although the system of eqs. (30) (abbreviated to (33)) does not admit a direct solution, the number of equations as well as the number of unknown functions being infinite, it seems to me very interesting that the solution to the multi-dimensional problem is provided in principle by a system of equations whose coefficients have such simple meanings in three dimensions. Further, one realises that the charge distribution that corresponds to the solution (29) of the n-electron problem turns out to be the superposition of the distributions ρ_{kk} and ρ_{kl} that occur already in the hydrogen problem. The hope of interpreting and of *understanding* the multi-dimensional theory in three dimensions is thus strengthened.

$E_l = E_k$, are different from zero. The problem thus separates in the first approximation into a denumerable set of *finite* principal axes problems.[a]

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[a] Summary of the above: One can embark on the solution of the system of equations (33) by an approximation method. Positing (34), the *constants* E and c_l have to satisfy the system (35) of ordinary linear homogeneous equations, whose number as well as that of the unknown constants, however, is infinite. It is only by assuming all coefficients v_{kl} to be *small* that one can conclude that E has to be very close to one of the values E_l, for instance E_k, and that $[c_l]$ approximately vanishes, unless E_l is equal to E_k. Since there is only a finite number of E_l that coincide with E_k, the problem reduces in the first approximation to a problem of a finite number of 'principal axes', or rather to an infinity of such finite problems. — As a matter of fact, the equations (35) coincide with the problem of an infinite number of principal axes, which the Heisenberg-Born mechanics reduces to.

Discussion of Mr Schrödinger's report

Mr Schrödinger. — It would seem that my description in terms of a snapshot was not very fortunate, since it has been misunderstood. Perhaps the following explanation is clearer. The interpretation of Born is well-known, who takes $\psi\psi^* d\tau$ to be the probability for the system being in the volume element $d\tau$ of the configuration space. Distribute a very large number N of systems in the configuration space, taking the above probability as 'frequency function'. Imagine these systems as superposed in real space, dividing however by N the charge of each point mass in each system. In this way, in the limiting case where $N = \infty$ one obtains the wave mechanical picture of the system.

Mr Bohr. — You have said that from the charge distribution $\psi\psi^* d\tau$ and the classical laws you obtain the frequency and intensity of light, but do the remarks about difficulties you made later indicate that what you had obtained was not correct?

Mr Schrödinger. — The difficulty I mentioned is the following. If one expands the general solution as a series with respect to the eigenfunctions

$$\psi = \sum_k c_k \psi_k$$

and if one calculates the intensity of the radiation resulting from ψ_k and ψ_l together, one finds that it becomes proportional to $c_k^2 c_l^2$. However, according to the old theory, only the square of the amplitude corresponding to the 'initial level' should appear here; that of the 'final level' should be replaced by 1.

Mr Bohr. — Has Dirac not found the solution to the difficulty?

Mr Schrödinger. — Dirac's results are certainly very interesting and point the way toward a solution, if they do not contain it already. Only, we should first come to an understanding in physical terms [nous devrions d'abord nous entendre en langage physique]. I find it still impossible, for the time being, to see an answer to a physical question in the assertion that certain quantities obey a noncommutative algebra, especially when these quantities are meant to represent numbers of atoms. The relation between the continuous spatial densities, described earlier, and the observed intensities and polarisations of the spectral rays is [too natural][a] for me to deny all meaning to these densities only because some difficulties appear that are not yet resolved.

Mr Born. — It seems to me that interpreting the quantity $\psi\psi^*$ as a charge density leads to difficulties in the case of quadrupole moments. The latter in fact need to be taken into account in order to obtain the radiation, not only for theoretical reasons, but also for experimental reasons.

For brevity let us set

$$e^2\psi\psi^* = e^2|\psi|^2 = \Psi$$

and let us consider, for example, the case of two particles; Ψ becomes a function of x_1 and x_2, where for brevity x_1 stands for all the coordinates of the first particle; x_2 has a similar meaning. The electric density is then, according to Schrödinger,

$$\rho(x) = \int \Psi(x, x_2)dx_2 + \int \Psi(x_1, x)dx_1 \ .$$

In wave mechanics the quadrupole moment

$$\int\int x_1 x_2 \Psi(x_1, x_2)dx_1 dx_2$$

cannot, as far as I can tell, be expressed using the function $\rho(x)$. I would like to know how one can, in this case, reduce the radiation of the quadrupole to the motion of a charge distribution $\rho(x)$ in the usual three-dimensional space.

Mr Schrödinger. — I can assure you that the calculation of the dipole moments is perfecly correct and rigorous and that this objection by Mr Born is unfounded. Does the agreement between wave mechanics and matrix mechanics extend to the possible radiation of a quadrupole? That is a question I have not examined. Besides, we do not possess observations on this point that could allow us to use a possible disagreement between the two approaches to decide between them.

Mr Fowler asks for explanations regarding the method for solving the equations in the case of the many-electron problem.

Mr De Donder. — Equation (24) of Mr Schrödinger's report can be

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[a] The French here reads 'trop peu naturelle', which has the exact opposite meaning. The context would seem, however, to justify the amendment (*eds.*).

extended to the case in which the n charged particles are *different* and where the external actions as well as the interactions can be described, in spacetime, by a gravitational field [champ gravifique].[a] The quantum equation thus obtained is the sum of the quantum equations for the n particles taken separately, each of the equations being divided by the (rest) mass of the corresponding particle. Thus, for instance, the quantum equation for the nucleus will not enter if one assumes, as a first approximation, that the mass of the nucleus is infinitely large with respect to that of an electron.

When there is interaction, the problem is much more complex. One can, as Mr Schrödinger indicates, consider the action of the nucleus as an *external* action acting on the electrons of the cloud [couronne], and the (electrostatic) actions *between* the electrons in this cloud as a *perturbation*; but that is only a first approximation. In order to account for relativistic and electromagnetic effects I have assumed that the molecular systems have an *additive* character.[a] One can thus recover, as a special case, the above-mentioned method of quantisation by Schrödinger.

Mr Born. — In Göttingen we have embarked on a systematic calculation of the matrix elements that appear in perturbation theory, with the aim of collecting them in tables up to the principal quantum number 10. Part of these calculations, which are very extended, has already been done. My coworker Mr Biemüller has used them to calculate the lower terms of the helium atom according to the usual perturbation method up to perturbations of the second order. The agreement of the ground term with the empirical value, despite the defects of the procedure, is hardly worse than in the recently published paper by Kellner [*Zeitschr. f. Phys.*, **44** (1927), 91], who has applied a more precise method (Ritz's procedure).

Mr Lorentz. — Do you see the outcome of this long labour as satisfactory?

Mr Born. — The calculation has not attained yet the precision of the measurements. The calculations we have done applying the ordinary

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[a] Th. De Donder, L'équation fondamentale de la Chimie quantique, *Comptes Rendus Acad. Sci. Paris*, session of 10 October 1927, pp. 698–700. See esp. eq. (10).

[a] For more details, one can consult our note: 'L'équation de quantification des molécules comprenant n particules électrisées', published after this meeting, in the *Bull. Ac. R. Belg., Cl. des Sciences*, session of 5 November 1927.

perturbation method [méthode des perturbations ordinaires] consist of a series expansion with respect to the inverse of the nuclear charge Z, of the form

$$E = Z\left(a + \frac{b}{Z} + \frac{c}{Z^2} + \ldots\right) .$$

The three terms shown have been calculated. Nevertheless, in the case of helium ($Z = 2$) the precision is not yet as good as in the calculations done by Kellner using Ritz's approximation method.

Mr Lorentz. — But you hope however to improve your results.

Mr Born. — Yes, only the convergence of the series is very slow.

Mr Heisenberg. — On the subject of this approximation method, Mr Schrödinger says at the end of his report that the discussion he has given reinforces the hope that when our knowledge will be deeper it will be possible to explain and to understand in three dimensions the results provided by the multi-dimensional theory. I see nothing in Mr Schrödinger's calculations that would justify this hope. What Mr Schrödinger does in his very beautiful approximation method, is to replace the n-dimensional differential equations by an infinity of linear equations. That reduces the problem, as Mr Schrödinger himself states, to a problem with ordinary matrices, in which the coefficients can be interpreted in three-dimensional space. The equations are thus 'three-dimensional' exactly in the same sense as in the usual matrix theory. It thus seems to me that, in the classical sense, we are just as far from understanding the theory in three dimensions as we are in the matrix theory.

Mr Schrödinger. — I would not know how to express more precisely my hope of a possible formulation in a three-dimensional space. Besides, I do not believe that one would obtain simpler calculational methods in this way, and it is probable that one will always do calculations using the multi-dimensional wave equation. But then one will be able to grasp its physical meaning better. I am not precisely searching for a three-dimensional partial differential equation. Such a simple formulation is surely impossible. If I am not satisfied with the current state of the problem, it is because I do not understand yet the physical meaning of its solution.

What Mr Heisenberg has said is mathematically unexceptionable, but the point in question is that of the physical interpretation. This is indispensable for the further development of the theory. Now, this development is necessary. For one must agree that all current ways of formulating the results of the new quantum mechanics only correspond to the classical mechanics of actions at a distance. As soon as light crossing times become relevant in the system, the new mechanics fails, because the classical potential energy function no longer exists.

Allow me, to show that my hope of achieving a three-dimensional conception is not quite utopian, to recall what Mr Fowler has told us on the topic of Mr Hartree's approximation method.[a] It is true that this method abstracts from what one calls the 'exchange terms' (which correspond, for instance, to the distance between the ortho and para terms of neutral helium). But, abstracting from that, it already achieves the three-dimensional aim I tend to. Should one declare *a priori* impossible that Hartree's method might be modified or developed in such a way as to take into account the exchange terms while working with a satisfactory three-dimensional model?

MR BORN. — Regarding the question of knowing whether it is possible to describe a many-electron problem by a field equation in three dimensions, I would like to point out the following. The number of quantum numbers of an atom rises by three with each additional electron; it is thus equal to $3n$ for n electrons. It seems doubtful that there should be an ordinary, three-dimensional eigenvalue problem, whose eigenvalues have a range of size ∞^{3n} [dont la valeur caractéristique ait une multitude de ∞^{3n} dimensions].[b] Instead, it follows from recent papers by Dirac and Jordan[c] that one can build on a three-dimensional oscillation equation if one considers the eigenfunction itself not as an ordinary number, but as one of Dirac's q-numbers, that is, if one quantises again its amplitude as a function of time. An n-quanta oscillation with this amplitude then yields together with the three spatial quantum numbers the necessary range [multitude] of quantum numbers. From this point of view the number of electrons in a system appears itself as a quantum number, that is, the electrons themselves appear as discontinuities of the same nature as the

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[a] See the discussion after Bragg's report, p. 318 (*eds.*).

[b] The French text here appears to make little sense, but Born is possibly referring to the dimension of the space of solutions (*eds.*).

[c] Cf. section IV of Born and Heisenberg's report (*eds.*).

stationary states.

Mr Schrödinger. — Precisely the structure of the periodic system is already contained in the physics [mécanique] of the three-dimensional hydrogen problem. The degrees of degeneracy 1, 4, 9, 16, etc., multiplied by 2, yield precisely the periodic numbers [nombres de périodes]. The factor 2 that I have just mentioned derives from the spin [giration (spin)]. From the point of view of wave mechanics, the apparently mysterious 'Pauli action' of the first two electrons on the third (which they prevent from also following an orbit with quantum number 1) means strictly speaking nothing other than the non-existence of a third eigenfunction with principal quantum number 1. This non-existence is precisely a property of the three-dimensional model, or of the three-dimensional equation. The multi-dimensional equation has too many eigenfunctions; it is this [elle] that makes the 'Pauli exclusion' (Pauliverbot) necessary to eliminate this defect.[a]

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[a] The French text refers to the four-dimensional equation ('l'équation à quatre dimensions') as having too many solutions. This reading could be correct, in the sense that the exclusion principle was first introduced in the context of the relativistic (four-dimensional) Bohr-Sommerfeld theory of the atom, but the above reading seems much more natural in context. Note that Schrödinger throughout his report uses 'vierdimensional' and 'vieldimensional', which could be easily confused, for 'four-dimensional' and 'many-dimensional', respectively (*eds.*).

Notes to the translation

1 Here and in the following, the French edition omits some italics, which are quite characteristic of Schrödinger's writing style and which we tacitly restore.
2 [Energiegerade] — [série des énergies]
3 Bracket added in the French edition.
4 [als Beschreibung] — [comme la définition]
5 [Vollständigkeit] — [perfection]
6 [freilich] — [évidemment]
7 Printed as a footnote in the French edition.
8 [sie nicht .... beschreibe] — [qu'ils ne décrivent pas]
9 [Einzelsystem] — [système déterminé]
10 This clause is omitted in the French edition.
11 [von anderer Seite vertreten] — [défendue par d'autres]. Footnote only in the French edition.
12 [aus einer Anzahl] — [d'un grand nombre]
13 [Durch die eben besprochene Deutung] — [Ainsi que nous venons de le voir]
14 The equation number is missing in the French edition, and the following sentence is printed as a footnote.
15 [es *gebe*] — [sont données]
16 [weitgehend] — [tout à fait]
17 [*Verständnis] — [interprétation]*
18 [*Verständnis*] — [signification]
19 No exclamation mark in the French edition.
20 [innere] — [intimes]
21 [eine prinzipielle] — [essentielle]
22 No exclamation mark in the French edition.
23 Printed as a footnote in the French edition.
24 In the French edition this equation number is given to the following equation (unnumbered in the typescript).
25 Bracket printed as a footnote in the French edition, with the addition: 'Δ stands for the Laplacian'.
26 [$\frac{e\varphi}{h}$] — [$e\frac{e\varphi}{h}$]
27 [die Ladungsdichte aus $\psi\psi^*$] — [la densité de charge $\psi\psi^*$]
28 [Betrag] — [valeur]
29 Bracket printed as a footnote in the French edition.
30 The rest of the bracket is printed as a footnote in the French edition.
31 Both typescript and French edition give '365' as page number.
32 The French edition adds this to the footnote.
33 [treten darin in ψ und seinen ersten Ableitungen quadratische Formen auf] — [y figurent dans ψ et ses premières dérivées des formes quadratiques]
34 Footnote only in the French edition.
35 [*bewegend*] — [*par le mouvement*]
36 [allerdings] — [certainement]
37 [(18)] — [(8)]
38 The French edition omits the inverted commas.
39 The equation number is missing in the printed volume.
40 [$3n$-dimensional] — [tridimensionelle]

41 Misprint in the French edition: the E_k are not in the exponent.
42 [komplet[t]e] — [complexe]
43 [dass für die Abhängigkeit der a von der Zeit folgende Forderungen bestehen] — [que pour que a dépende du temps les conditions suivantes doivent être satisfaites]
44 Two misprints in the French edition: the E_k are not in the exponent, and the k_i run to n.
45 Printed as a footnote in the French edition.
46 [unseres Ansatzes] — [de notre expression fondamentale]
47 [den *einzelnen*] — [les *diverses*]
48 Bracket printed as footnote in the French edition.
49 [bestimmen die Lösung exact] — [déterminent la solution]
50 Misprint in the French edition: '$\psi(k_1, k_2, \ldots, k_n)$'.
51 Both the typescript and the French edition read 'c_l' and 'c_k' instead of 'a_l' and 'a_k'.
52 Again, both the typescript and the French edition read 'c_k' instead of 'a_k'.

General discussion of the new ideas presented[a]

CAUSALITY, DETERMINISM, PROBABILITY

MR LORENTZ. — I should like to draw attention to the difficulties one encounters in the old theories.

We wish to make a representation of the phenomena, to form an image of them in our minds. Until now, we have always wanted to form these images by means of the ordinary notions of time and space. These notions are perhaps innate; in any case, they have developed from our personal experience, by our daily observations. For me, these notions are clear and I confess that I should be unable to imagine physics without these notions. The image that I wish to form of phenomena must be absolutely sharp and definite, and it seems to me that we can form such an image only in the framework of space and time.

For me, an electron is a corpuscle that, at a given instant, is present at a definite point in space, and if I had the idea that at a following moment the corpuscle is present somewhere else, I must think of its trajectory, which is a line in space. And if the electron encounters an atom and penetrates it, and after several incidents leaves the atom, I make up a theory in which the electron preserves its individuality; that is to say, I imagine a line following which the electron passes through the atom.

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[a] As mentioned in section 1.6, the Bohr archives contain a copy of the galley proofs of the general discussion, dated 1 June 1928.[1] A few of the contributions in these proofs seem to have been still largely unedited: they contain some gaps and incomplete sentences, some more colloquial formulations, and in at least one case a sentence that was dropped from the published volume. We reproduce in endnotes the most substantial examples of these alternative versions. For most of the discussion contributions by Dirac, we have followed his manuscript version.[2] For Bohr's discussion contributions, we have used material from Bohr (1985) and from notes taken by Richardson[3] (also mentioned in section 1.6). See our notes for further details (*eds.*).

Obviously, such a theory may be very difficult to develop, but *a priori* it does not seem to me impossible.

I imagine that, in the new theory, one still has electrons. It is of course possible that in the new theory, once it is well-developed, one will have to suppose that the electrons undergo transformations. I happily concede that the electron may dissolve into a cloud. But then I would try to discover on which occasion this transformation occurs. If one wished to forbid me such an enquiry by invoking a principle, that would trouble me very much. It seems to me that one may always hope one will do later that which we cannot yet do at the moment. Even if one abandons the old ideas, one may always preserve the old classifications [dénominations]. I should like to preserve this ideal of the past, to describe everything that happens in the world with distinct images. I am ready to accept other theories, on condition that one is able to re-express them in terms of clear and distinct images.

For my part, despite not having yet become familiar with the new ideas that I now hear expressed,[a] I could visualise these ideas thus. Let us take the case of an electron that encounters an atom; let us suppose that the electron leaves the atom and that at the same time there is emission of a light quantum. One must consider, in the first place, the systems of waves that correspond to the electron and to the atom before the collision. After the collision, we will have new systems of waves. These systems of waves can be described by a function ψ defined in a space with a large number of dimensions and satisfying a differential equation. The new wave mechanics will work with this equation and will determine the function ψ before and after the collision.

Now, there are phenomena that teach us that there is something else in addition to the waves, namely corpuscles; one can, for example, perform an experiment with a Faraday cylinder; one must then take into account the individuality of the electrons and also of the photons. I think I would find that, to explain the phenomena, it suffices to assume that the expression $\psi\psi^*$ gives the probability that the electrons and the photons exist in a given volume; that would suffice to explain the experiments. But the examples given by Mr Heisenberg teach me that I will have thus attained everything that experiment allows me to attain. However, I think that this notion of probability should be placed at

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[a] In fact, Lorentz had followed the recent developments rather closely. In particular, he had corresponded extensively with Ehrenfest and with Schrödinger, and had even delivered seminars and lectures on wave mechanics and on matrix mechanics at Leiden, Cornell and Caltech. See section 1.3 (*eds.*).

the end, and as a conclusion, of theoretical considerations, and not as an *a priori* axiom, though I may well admit that this indeterminacy corresponds to experimental possibilities. I would always be able to keep my deterministic faith for the fundamental phenomena, of which I have not spoken. Could a deeper mind not be aware of the motions of these electrons? Could one not keep determinism by making it an object of belief? Must one necessarily elevate indeterminism to a principle?

MR BOHR expounds his point of view with respect to the problems of quantum theory.

The original published proceedings add '(see the preceding article)'. In the proceedings, the article preceding the general discussion is a French translation of the German version of Bohr's Como lecture (Bohr 1928) (published in *Naturwissenschaften*). As described in section 1.6, this article was included at Bohr's request, to replace his remarks made at this point in the general discussion. (In our translation of the proceedings, we have omitted this well-known article.)

The extant notes relating to Bohr's remarks at this point are particularly fragmentary. Kalckar's introduction to volume 6 of Bohr's *Collected Works* (Bohr 1985) describes the corresponding part of notes (taken by Kramers and by Verschaffelt) in the Bohr archives as too incomplete to warrant reproduction in that volume, but provides the following summary and comparison with the printed versions of the Como lecture: 'The notes cover the wave-corpuscle aspects of light and matter (corresponding to the first sections of the printed lecture). The γ-ray microscope is analysed, although the notes are somewhat incomplete here (as in many other places), and the rôle of the finite wave trains is discussed in connection with the momentum measurement through the Doppler effect (as in the printed versions). After some questions .... Bohr continues by discussing the significance of the phase and comments on the Stern-Gerlach experiment and the inobservability of the phase in a stationary state .... ' (Bohr 1985, p. 37).

Further details of what Bohr said at this point may be obtained from notes on the general discussion taken by Richardson.[4] Below, we reproduce the relevant parts of these notes, and comment on their relation to Bohr's paper translated in the proceedings.

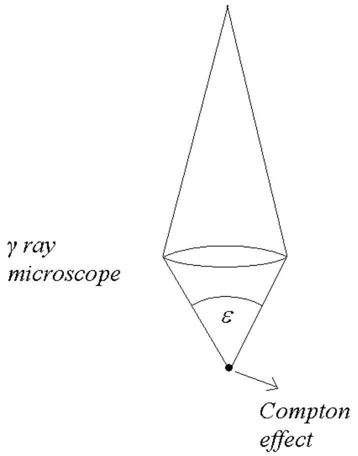
The first part of Richardson's notes relating to Bohr reads as follows:

$$E = h\nu \qquad e^{i2\pi(\tau_x x+\tau_y y+\tau_z z-\nu t)}$$

$$p = h\tau$$

Int[er]f[eren]ce. $?h \to \infty$ [?]

$$\left.\begin{array}{lcl} \Delta x \Delta \tau_x & \sim & 1 \\ \Delta t \Delta \nu & \sim & 1 \end{array}\right\} \qquad \begin{array}{lcl} \Delta x \Delta p_x & \sim & h \\ \Delta t \Delta E & \sim & h \end{array}$$



$$\Delta x \sim \frac{\lambda}{\varepsilon}$$

$$\Delta p_x \sim \varepsilon \cdot h\tau$$

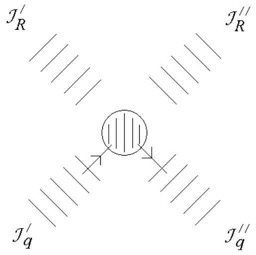
$$\Delta p_x \Delta x \sim h.$$

Fig. A.

This corresponds to part of section 2 of Bohr's paper translated in the proceedings. There Bohr introduces the concepts of energy and momentum for plane waves, and the idea that waves of limited extent in spacetime are obtained through the 'interference' (that is, superposition) of different plane waves, the resulting waves satisfying (at best) the given relations. (As a consequence, a group of waves has no well-defined phase, a point Bohr takes up again below.) This is used to justify Bohr's idea of complementarity between a causal picture (in the sense of energy-momentum conservation for elementary processes) and a spacetime picture.

Richardson's notes then continue as shown in Fig. A. The γ-ray microscope is discussed in section 3 of Bohr (1928) (the section on measurement, which also discusses momentum measurements based on the Doppler effect). Bohr appears to have inserted a discussion of these experiments as an illustration of the uncertainty-type relations above.

The next part of Richardson's notes returns to section 2 of the paper, and is reproduced in Fig. B. This corresponds in fact to the subsequent paragraphs of section 2, in which Bohr applies the notion of complementarity to resolve the perceived paradoxes related to the scattering of radiation by free electrons (note the extended — as opposed to pointlike — region of scattering in the diagram, and see Bohr's contribution to the discussion of Compton's report, p. 60) as well as the perceived paradoxes related to collisions (cf. section 3.4.2).



$\mathcal{J}'_R + \mathcal{J}'_q = \mathcal{J}''_R + \mathcal{J}''_q$

Energy & momentum p[rinci]ples. Must have same uncertainty on both sides of this =^{n} [equation].

Important that formulae do not contain the mass. De Donder says also in relativity theory.

Fig. B.

Possibly, $\mathcal{I}$ stands for 'Impuls' (that is, momentum), R for radiation, q for charge.

The next part of Richardson's notes, shown in Fig. C, instead relates to part of section 6 of Bohr's paper (sections 4 and 5 of the paper are, respectively, a review of the correspondence principle and of matrix mechanics, and a discussion and critique of wave mechanics). In section 6 of the published paper, Bohr raises the following puzzle. According to Bohr, in any observation that distinguishes between different stationary states one has to disregard the past history of the atom, but, paradoxically, the theory assigns a phase to a stationary state. However, since the system will not be strictly isolated, one will work with a group of waves, which (as mentioned in section 2) has no well-defined phase. Bohr then illustrates this with the Stern-Gerlach experiment. The condition for distinguishability of the eigenstates of the hydrogen atom is that the angular spreading of the beam should be greater than that given by diffraction at the slit ($\varepsilon > \alpha$), which translates into the time-energy uncertainty relation. As Bohr mentions, Heisenberg (1927) uses this as an illustration of the uncertainty relation, while Bohr uses it as an illustration of how knowledge of the phase is lost. (This section also discusses the limit of high quantum numbers.)

The final section 7 of the paper ('The problem of elementary particles') has no parallel in Richardson's notes. The part of the notes relating to Bohr's remarks at this point concludes instead with the following (explicitly labelled 'Bohr'):

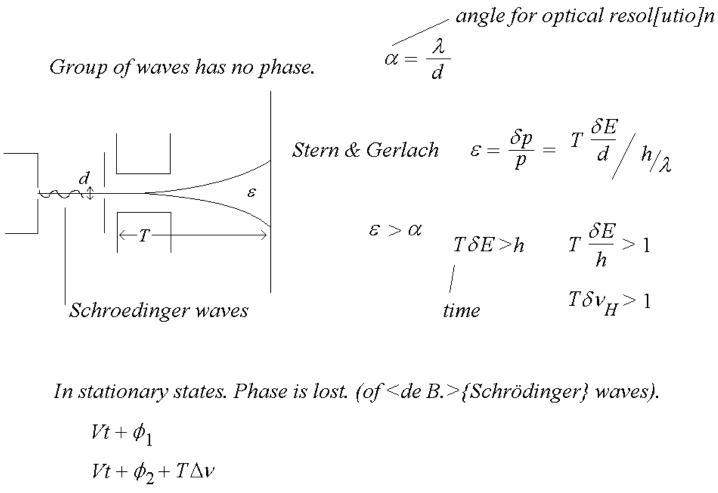


Fig. C.

1. [blank]
2. Stationary states, past lost ∵ [because] phase indetermination — Stern & Gerlach's Exp[erimen]t.
3. Schroedinger's ψ, prob[abilit]y of electron at a given place at a given time [?], uncertainty $\Delta\nu\Delta t \sim 1$

$$\gamma \times \frac{v}{c}$$

Mr Brillouin. — Mr Bohr insists on the uncertainty of simultaneous measurements of position and momentum; his point of view is closely connected to the notion of *cells in phase space* introduced by Planck a very long time ago. Planck assumed that if the representative point of a system is in a cell (of size $\Delta p\Delta q = h$) one cannot distinguish it from another point in the same cell. The examples brought by Mr Bohr aptly make precise the physical meaning of this quite abstract notion.

Mr De Donder. — The considerations that Mr Bohr has just developed are, I think, in close relation with the following fact: in the

Einsteinian Gravitation[a] of a continuous system or of a pointlike system, there appear not the masses and charges of the particles, but entities $\tau^{(m)}$ and $\tau^{(e)}$ in *four* dimensions; note that these *generalised* masses and charges, localised in spacetime, *are conserved* along their worldlines.

Mr Born. — Mr Einstein has considered the following problem: A radioactive sample emits α-particles in all directions; these are made visible by the method of the Wilson cloud [chamber]. Now, if one associates a spherical wave with each emission process, how can one understand that the track of each α particle appears as a (very nearly) straight line? In other words: how can the corpuscular character of the phenomenon be reconciled here with the representation by waves?

To do this, one must appeal to the notion of 'reduction of the probability packet' developed by Heisenberg.[a] The description of the emission by a spherical wave is valid only for as long as one does not observe ionisation; as soon as such ionisation is shown by the appearance of cloud droplets, in order to describe what happens afterwards one must 'reduce' the wave packet in the immediate vicinity of the drops. One thus obtains a wave packet in the form of a ray, which corresponds to the corpuscular character of the phenomenon.

Mr Pauli[b] has asked me if it is not possible to describe the process without the reduction of wave packets, by resorting to a multi-dimensional space, whose number of dimensions is three times the number of all the particles present (α-particles and atoms hit by the radiation).

This is in fact possible and can even be represented in a very anschaulich manner [d'une manière fort intuitive] by means of an appropriate simplification, but this does not lead us further as regards the fundamental questions. Nevertheless, I should like to present this case here as an example of the multi-dimensional treatment of such problems. I assume, for simplicity, that there are only two atoms that may be hit. One then has to distinguish two cases: either the two atoms 1 and 2 lie on the same ray starting from the origin (the place where the preparation is),

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[a] Th. De Donder, *Théorie des champs gravifiques* (*Mémorial des sciences mathématiques*, part 14, Paris, 1926). See esp. equations (184), (184´) and (188), (188´). One can also consult our lectures: *The Mathematical Theory of Relativity* (Massachusetts Institute of Technology), Cambridge, Mass., 1927. See esp. equations (23), (24) and (28), (29).

[a] Born is referring here in particular to Heisenberg's uncertainty paper (Heisenberg 1927) (*eds.*).

[b] Cf. Pauli's letter to Bohr, 17 October 1927, discussed in section 6.2.1 (*eds.*).

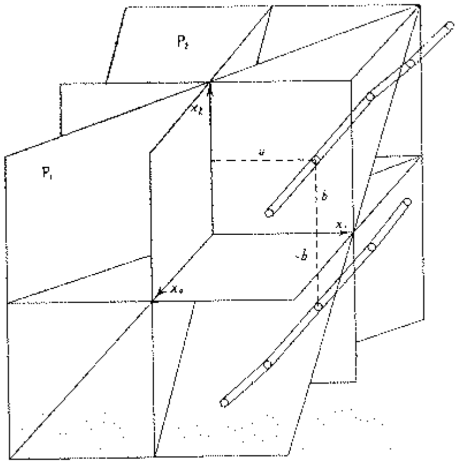

Fig. 1.

or they do not lie on the same ray. If we represent by ε the probability that an atom will be hit, we have the following probability diagram:[a]

I. The points 1 and 2 are located on the same ray starting from the origin.

Number of particles hit	Probability
0	$1-\varepsilon$
1	0
2	ε

---

[a] In the following tables, the probability for the number of particles hit to equal 1 should be read as the probability for *each case* in which the number of particles hit equals 1 (*eds.*).

II. The points 1 and 2 are not on the same ray.

Number of particles hit	Probability
0	$1-2\varepsilon$
1	ε
2	0

This is how one should express the probability of events in the case of rectilinear propagation.

To make possible a graphical representation of the phenomenon, we will simplify it further by assuming that all the motions take place following only a single straight line, the axis x. We must then distinguish the two cases where the atoms lie on the same side and on either side of the origin. The corresponding probabilities are the following:

I. The points 1 and 2 are located on the same side.

Number of particles hit	Probability
0	$\frac{1}{2}$
1	0
2	$\frac{1}{2}$

II. The points 1 and 2 are located on different sides.

Number of particles hit	Probability
0	0
1	$\frac{1}{2}$
2	0

Now, these relations can be represented by the motion of a wave packet in a space with three dimensions x_0, x_1, x_2. To the initial state there corresponds:

In case I, the point $x_0 = 0, \quad x_1 = a \quad x_2 = b$
In case II, the point $x_0 = 0, \quad x_1 = a \quad x_2 = -b$

where a and b are positive numbers. The wave packet at first fills the space surrounding these points and subsequently moves parallel to the axis x_0, dividing itself into two packets of the same size going in opposite directions. Collisions are produced when $x_0 = x_1$ or $x_0 = x_2$, that is to say, on two planes of which one, P_1, is parallel to the axis x_2 and cuts the plane x_0x_1 following the bisector of the positive quadrant, while the second, P_2, is parallel to the axis x_1 and cuts the plane x_0x_2 following the bisector of the positive quadrant. As soon as the wave packet strikes the plane P_1, its trajectory receives a small kink in the direction x_1; as soon as it strikes P_2 the trajectory receives a kink in the direction x_2 (Fig. 1).

Now, one immediately sees in the figure that the upper part of the wave packet, which corresponds to case I, strikes the planes P_1, P_2 on the same side of the plane x_1x_2, while the lower part strikes them[5] on different sides. The figure then gives an anschaulich representation of the cases indicated in the above diagram. It allows us to recognise immediately whether, for a given size of wave packet, a given state, that is to say a given point x_0, x_1, x_2, can be hit or not.

To the 'reduction' of the wave packet corresponds the choice of one of the two directions of propagation $+x_0$, $-x_0$, which one must take as soon as it is established that one of the two points 1 and 2 is hit, that is to say, that the trajectory of the packet has received a kink.

This example serves only to make clear that a complete description of the processes taking place in a system composed of several molecules is possible only in a space of several dimensions.

MR EINSTEIN.[a] — Despite being conscious of the fact that I have not entered deeply enough into the essence of quantum mechanics, nevertheless I want to present here some general remarks.[b]

One can take two positions towards the theory with respect to its postulated domain of validity, which I wish to characterise with the aid of a simple example.

Let S be a screen provided with a small opening O (Fig. 2), and P a hemispherical photographic film of large radius. Electrons impinge on S in the direction of the arrows. Some of these go through O, and because of the smallness of O and the speed of the particles, are dispersed uniformly over the directions of the hemisphere, and act on the film.

Both ways of conceiving the theory now have the following in common. There are de Broglie waves, which impinge approximately normally on S and are diffracted at O. Behind S there are spherical waves, which reach the screen P and whose intensity at P is responsible [massgebend] for what happens at P.[c]

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[a] The extant manuscript in the Einstein archives[6] consists of the first four paragraphs only, which we have translated here (footnoting significant differences from the published French) (*eds.*).

[b] The published French has: 'I must apologise for not having gone deeply into quantum mechanics. I should nevertheless want to make some general remarks' (*eds.*).

[c] In the published French, the German expression 'ist massgebend' is misrendered

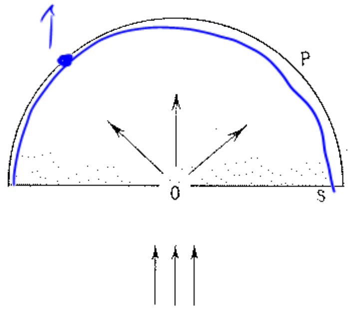

Fig. 2.

We can now characterise the two points of view as follows.

1. *Conception I.* — The de Broglie-Schrödinger waves do not correspond to a single electron, but to a cloud of electrons extended in space. The theory gives no information about individual processes, but only about the ensemble of an infinity of elementary processes.

2. *Conception II.* — The theory claims to be a complete theory of individual processes. Each particle directed towards the screen, as far as can be determined by its position and speed, is described by a packet of de Broglie-Schrödinger waves of short wavelength and small angular width. This wave packet is diffracted and, after diffraction, partly reaches the film P in a state of resolution [un état de résolution].

According to the first, purely statistical, point of view $|\psi|^2$ expresses the probability that there exists at the point considered *a particular* particle of the cloud, for example at a given point on the screen.

According to the second, $|\psi|^2$ expresses the probability that at a given instant *the same* particle is present at a given point (for example on the screen). Here, the theory refers to an individual process and claims to describe everything that is governed by laws.

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as 'donne la mesure' [gives the measure] instead of as 'is responsible'. This is of some significance for the interpretation of Einstein's remarks as a form of the later EPR argument; see section 7.1 (*eds.*).

The second conception goes further than the first, in the sense that all the information resulting from I results also from the theory by virtue of II, but the converse is not true.[a] It is only by virtue of II that the theory contains the consequence that the conservation laws are valid for the elementary process; it is only from II that the theory can derive the result of the experiment of Geiger and Bothe, and can explain the fact that in the Wilson [cloud] chamber the droplets stemming from an α-particle are situated very nearly on continuous lines.

But on the other hand, I have objections to make to conception II. The scattered wave directed towards P does not show any privileged direction. If $|\psi|^2$ were simply regarded as the probability that at a certain point a given particle is found at a given time, it could happen that *the same* elementary process produces an action *in two or several* places on the screen. But the interpretation, according to which $|\psi|^2$ expresses the probability that *this* particle is found at a given point, assumes an entirely peculiar mechanism of action at a distance, which prevents the wave continuously distributed in space from producing an action in *two* places on the screen.

In my opinion, one can remove this objection only in the following way, that one does not describe the process solely by the Schrödinger wave, but that at the same time one localises the particle during the propagation. I think that Mr de Broglie is right to search in this direction. If one works solely with the Schrödinger waves, interpretation II of $|\psi|^2$ implies to my mind a contradiction with the postulate of relativity.

I should also like to point out briefly two arguments which seem to me to speak against the point of view II. This [view] is essentially tied to a multi-dimensional representation (configuration space), since only this mode of representation makes possible the interpretation of $|\psi|^2$ peculiar to conception II. Now, it seems to me that objections of principle are opposed to this multi-dimensional representation. In this representation, indeed, two configurations of a system that are distinguished only by the permutation of two particles of the same species are represented by two different points (in configuration space), which is not in accord with the new results in statistics. Furthermore, the feature of forces of acting only at small *spatial* distances finds a less natural expression in configuration space than in the space of three or four dimensions.

MR BOHR.[a] — I feel myself in a very difficult position because I don't

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[a] The French has 'I' and 'II' exchanged in this sentence, which is illogical (*eds.*).

[a] These remarks by Bohr do not appear in the published French. We have reproduced

understand what precisely is the point which Einstein wants to [make]. No doubt it is my fault.

....

As regards general problem I feel its difficulties. I would put problem in other way. I do not know what quantum mechanics is. I think we are dealing with some mathematical methods which are adequate for description of our experiments. Using a rigorous wave theory we are claiming something which the theory cannot possibly give. [We must realise] that we are away from that state where we could hope of describing things on classical theories. Understand same view is held by Born and Heisenberg. I think that we actually just try to meet, as in all other theories, some requirements of nature, but difficulty is that we must use words which remind of older theories. The whole foundation for causal spacetime description is taken away by quantum theory, for it is based on assumption of observations without interference. .... excluding interference means exclusion of experiment and the whole meaning of space and time observation .... because we [have] interaction [between object and measuring instrument] and thereby we put us on a quite different standpoint than we thought we could take in classical theories. If we speak of observations we play with a statistical problem. There are certain features complementary to the wave pictures (existence of individuals). ....

....

The saying that spacetime is an abstraction might seem a philosophical triviality but nature reminds us that we are dealing with something of practical interest. Depends on how I consider theory. I may not have understood, but I think the whole thing lies [therein that the] theory is nothing else [but] a tool for meeting our requirements and I think it does.

Mr Lorentz. — To represent the motion of a system of n material points, one can of course make use of a space of 3 dimensions with n points or of a space of $3n$ dimensions where the systems will be represented by a single point. This must amount to exactly the same thing; there can be no fundamental difference. It is merely a question of

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them from Bohr's *Collected Works*, vol. 6 (Bohr 1985, p. 103), which contains a reconstruction of Bohr's remarks from notes by Verschaffelt (held in the Bohr archive). The tentative interpolations in square brackets are by the editor of Bohr (1985), J. Kalckar (*eds.*).

knowing which of the two representations is the most suitable, which is the most convenient.

But I understand that there are cases where the matter is difficult. If one has a representation in a space of $3n$ dimensions, one will be able to return to a space of 3 dimensions only if one can reasonably separate the $3n$ coordinates into n groups of 3, each corresponding to a point, and I could imagine that there may be cases where that is neither natural nor simple. But, after all, it certainly seems to me that all this concerns the form rather than the substance of the theory.

Mr Pauli. — I am wholly of the same opinion as Mr Bohr, when he says that the introduction of a space with several dimensions is only a technical means of formulating mathematically the laws of mutual action between several particles, actions which certainly do not allow themselves to be described simply, in the ordinary way, in space and time. It may perfectly well be that this technical means may one day be replaced by another, in the following fashion. By Dirac's method one can, for example, quantise the characteristic vibrations of a cavity filled with blackbody radiation, and introduce a function ψ depending on the amplitudes of these characteristic vibrations of unlimited number. One can similarly use, as do Jordan and Klein, the amplitudes of ordinary four-dimensional material waves as arguments of a multi-dimensional function ϕ. This gives, in the language of the corpuscular picture, the probability that at a given instant the numbers of particles of each species present, which have certain kinematical properties (given position or momentum), take certain values. This procedure also has the advantage that the defect of the ordinary multi-dimensional method, of which Mr Einstein has spoken and which appears when one permutes two particles of the same species, no longer exists. As Jordan and Klein have shown, making suitable assumptions concerning the equations that this function ϕ of the amplitudes of material waves in ordinary space must satisfy,[7] one arrives exactly at the same results as by basing oneself on Schrödinger's multi-dimensional theory.

To sum up, I wish then to say that Bohr's point of view, according to which the properties of physical objects of being defined and of being describable in space and time are complementary, seems to be more general than a special technical means. But, independently of such a means, one can, according to this idea, declare in any case that the mutual actions of several particles certainly cannot be described in the ordinary manner in space and time.

To make clear the state of things of which I have just spoken, allow me to give a special example. Imagine two hydrogen atoms in their ground state at a great distance from each other, and suppose one asks for their energy of mutual action. Each of the two atoms has a perfectly isotropic distribution of charge, is neutral as a whole, and does not yet emit radiation. According to the ordinary description of the mutual action of the atoms in space and time, one should then expect that such a mutual action does not exist when the distance between the two neutral spheres is so great that no notable interpenetration takes place between their charge clouds. But when one treats the same question by the multi-dimensional method, the result is quite different, and in accordance with experiment.

The classical analogy to this last result would be the following: Imagine inside each atom a classical oscillator whose moment p varies periodically. This moment produces a field at the location of the other atom whose periodically variable intensity is of order $\mathcal{E} \sim \frac{p}{r^3}$, where r is the distance between the two atoms. When two of these oscillators act on each other, a polarisation occurs with the following potential energy, corresponding to an attractive force between the atoms,

$$-\frac{1}{2}\alpha\mathcal{E}^2 \sim \frac{1}{2}\alpha p^2 \frac{1}{r^6} \ ,$$

where α represents the polarisability of the atom.

In speaking of these oscillators, I only wanted to point out a classical analogy with the effect that one obtains as a result of multi-dimensional wave mechanics. I had found this result by means of matrices, but Wang has derived it directly from the wave equation in several dimensions. In a paper by Heitler and London, which is likewise concerned with this problem, the authors have lost sight of the fact that, precisely for a large distance between the atoms, the contribution of polarisation effects to the energy of mutual action, a contribution which they have neglected, outweighs in order of magnitude the effects they have calculated.

Mr Dirac.[a] — I should like to express my ideas on a few questions.

The first is the one that has just been discussed and I have not much to add to this discussion. I shall just mention the explanation that the

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[a] Here we mostly follow the English version from Dirac's manuscript.[8] (The French translation may have been done from a typescript or fairer copy.) We generally follow the French paragraphing, and we uniformise Dirac's notation. Interesting variants, cancellations and additions will be noted, as will significant deviations from the published French (*eds.*).

quantum theory would give of Bothe's experiment.[9] The difficulty arises from[10] the inadequacy of the 3-dimensional wave picture. This picture cannot distinguish between the case when there is a probability p of a light-quant being in a certain small volume, and the case when there is a probability $\frac{1}{2}p$ of two light-quanta being in the volume, and no probability for only one. But the wave function in many-dimensional space does distinguish between these cases. The theory of Bothe's experiment in many-dimensional space would show that, while there is a certain probability for a light-quantum appearing in one or the other of the counting chambers, there is no probability of two appearing simultaneously.

At present the general theory of the wave function in many-dimensional space necessarily involves the abandonment of relativity.[11] One might, perhaps, be able to bring relativity into the general quantum theory in the way Pauli has mentioned of quantising 3-dimensional waves, but this would not lead to greater Anschaulichkeit[12] in the explanation of results such as Bothe's.

I shall now show how Schrödinger's expression for the electric density appears naturally in the matrix theory. This will show the exact signification of this density and the limitations which must be imposed on its use. Consider an electron moving in an arbitrary field, such as that of an H atom. Its coordinates x, y, z will be matrices. Divide the space up into a number of cells, and form that function of x, y, z that is equal to 1 when the electron is in a given cell and 0 otherwise. This function of the matrices x, y, z will also be a matrix.[a] There is one such matrix for each cell whose matrix elements will be functions of the coordinates a, b, c of the cell, so that it can be written $A(a, b, c)$.

Each of these matrices represents a quantity that if measured experimentally must have either the value 0 or 1. Hence each of these matrices has the characteristic values 0 and 1 and no others. If one takes the two matrices $A(a, b, c)$ and $A(a', b', c')$, one sees that they must commute,[13] since one can give a numerical value to both simultaneously; for example, if the electron is known to be in the cell a, b, c, it will certainly not be in the cell a', b', c', so that if one gives the numerical value 1 to $A(a, b, c)$, one must at the same time give the numerical value 0 to $A(a', b', c')$.

We can transform each of the matrices A into a diagonal matrix A^*

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[a] The published version has: 'Divide the space up into a large number of small cells, and consider the function of three variables ξ, η, ζ that is equal to 1 when the point ξ, η, ζ is in a given cell and equal to 0 when the point is elsewhere. This function, applied to the matrices x, y, z, gives another matrix' (*eds.*).

by a transformation[14] of the type

$$A^* = BAB^{-1}.$$

Since all the matrices $A(a, b, c)$ commute,[15] they can be transformed simultaneously into diagonal matrices by a transformation of this type. The diagonal elements of each matrix $A^*(a, b, c)$ are its characteristic values, which are the same as the characteristic values of $A(a, b, c)$, that is, 0 and 1.

Further, no two A^* matrices, such as $A^*(a, b, c)$ [and] $A^*(a', b', c')$, can both have 1 for the same diagonal element, as a simple argument shows that $A^*(a, b, c)+A^*(a', b', c')$ must also have only the characteristic values 0 and 1. We can without loss of generality assume that each A^* has just one diagonal element equal to 1 and all the others zero. By transforming back, by means of the formula

$$A(a, b, c) = B^{-1}A^*(a, b, c)B \ ,$$

we now find that the matrix elements of $A(a, b, c)$ are of the form

$$A(a, b, c)_{mn} = B_m^{-1}B_n \ ,$$

i.e. a function of the row multiplied by a function of the column.

It should be observed that the proof of this result is quite independent of equations of motion and quantum conditions. If we take these into account, we find that B_m^{-1} and B_n are apart from constants just Schrödinger's eigenfunctions $\bar{\psi}_m$ and ψ_n at the point a, b, c.

Thus Schrödinger's density function $\bar{\psi}_m(x, y, z)\psi_m(x, y, z)$ is a[16] diagonal element of the matrix A referring to a cell about the point x, y, z. The true quantum expression for the density is the whole matrix. Its diagonal elements give only the average density, and must not be used when the density is to be multiplied by a dynamical variable represented by a matrix.

I should now like to express my views on determinism and the nature of the numbers appearing in the calculations of the quantum theory, as they appear to me after thinking over Mr Bohr's remarks of yesterday.[17] In the classical theory one starts from certain numbers describing completely the initial state of the system, and deduces other numbers that describe completely the final state. This deterministic theory applies only to an isolated system.

But, as Professor Bohr has pointed out, an isolated system is by definition unobservable. One can observe the system only by disturbing it and observing its reaction to the disturbance. Now since physics is concerned only with observable quantities the deterministic classical theory is untenable.[18]

In the quantum theory one also begins with certain numbers and deduces others from them. Let us inquire into the distinguishing characteristics[19] of these two sets of numbers. The disturbances that an experimenter applies to a system to observe it are directly under his control, and are acts of freewill by him. *It is only the numbers that describe these acts of freewill that can be taken as initial numbers for a calculation in the quantum theory.* Other numbers describing the initial state of the system are inherently unobservable, and do[20] not appear in the quantum theoretical treatment.

Let us now consider the final numbers obtained as the result of an experiment. It is essential that the result of an experiment shall be a permanent record. The numbers that describe such a result must help to not only describe the state of the world at the instant the experiment is ended, but also help to describe the state of the world at any subsequent time. These numbers describe what is common to all the events in a certain chain of causally connected events, extending indefinitely into the future.

Take as an example a Wilson cloud expansion experiment. The causal chain here consists of the formation of drops of water round ions, the scattering of light by these drops of water, and the action of this light on a photographic plate, where it leaves a permanent record. The numbers that form the result of the experiment describe all of the events in this chain equally well and help to describe the state of the world at any time after the chain began.

One could perhaps extend the chain further into the past.[21] In the example one could, perhaps, ascribe the formation of the ions to a β-particle, so that the result of the experiment would be numbers describing the track of a β-particle. In general one tries with the help of theoretical considerations to extend the chain as far back into the past as possible, in order that the numbers obtained as the result of the experiment may apply as directly as possible to the process under investigation.[22]

This view of the nature of the results of experiments fits in very well with the new quantum mechanics. According to quantum mechanics the state of the world at any time is describable by a wave function ψ, which normally varies according to a causal law, so that its initial value determines its value at any later time. It may however happen that at a certain time t_1, ψ can be expanded in the form

$$\psi = \sum_n c_n \psi_n \ ,$$

where the ψ_n's are wave functions of such a nature that they cannot interfere with one another at any time subsequent to t_1. If such is the case, then the world at times later than t_1 will be described not by ψ but by one of the ψ_n's. The particular ψ_n that it shall be must be regarded as chosen by nature.[23] One may say that nature chooses which ψ_n it is to be, as the only information given by the theory is that the probability of any ψ_n being chosen is $|c_n|^2$.[24] The value of the suffix n that labels the particular ψ_n chosen may be the result of an experiment, and the result of an experiment must always be such a number. It is a number describing an irrevocable choice of nature, which must affect the whole of the future course of events.[a]

As an example take the case of a simple collision problem. The wave packet representing the incident electron gets scattered in all directions. One must take for the wave function after the process not the whole scattered wave, but once again a wave packet moving in a definite direction. From the results of an experiment, by tracing back a chain of causally connected events one could determine in which direction the electron was scattered and one would thus infer that nature had chosen this direction. If, now, one arranged a mirror to reflect the electron wave scattered in one direction d_1 so as to make it interfere with the electron wave scattered in another direction d_2, one would not be able to distinguish between the case when the electron is scattered in the direction d_2 and when it is scattered in the direction d_1 and reflected back into d_2. One would then not be able to trace back the chain of causal events so far, and one would not be able to say that nature had chosen a direction as soon as the collision occurred, but only [that] at a later time nature chose where the electron should appear. The[25] interference between the ψ_n's compels nature to postpone her choice.

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[a] The last two sentences appear differently in the published version: 'The choice, once made, is irrevocable and will affect the whole future state of the world. The value of n chosen by nature can be determined by experiment and *the results of all experiments* are numbers describing such choices of nature'.
Dirac's notes contain a similar variant written in the margin: 'The value of n chosen by nature may be determined by experiment. The result of every experiment consists of numbers determining one of these choices of nature, and is permanent since such a choice is irrevocable and affects the whole future state of the world' (*eds.*).

Mr Bohr.[a] — Quite see that one must go into details of pictures, if one wants to control or illustrate general statements. I think still that you may simpler put it in my way. Just this distinction between observation and definition allows to let the quantum mechanics appear as generalisation. What does mean: get records which do not allow to work backwards. Even if we took all molecules in photographic plate one would have closed system. If we tell of a record we give up definition of plate. Whole point lies in that by observation we introduce something which does not allow to go on.

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Mr Born. — I should like to point out, with regard to the considerations of Mr Dirac, that they seem closely related to the ideas expressed in a paper by my collaborator J.[26] von Neumann, which will appear shortly. The author of this paper shows that quantum mechanics can be built up using the ordinary probability calculus, starting from a small number of formal hypotheses; the probability amplitudes and the law of their composition do not really play a role there.

Mr Kramers. — I think the most elegant way to arrive at the results of Mr Dirac's considerations is given to us by the methods he presented in his memoir in the *Proc. Roy. Soc.*, ser. *A*, vol. **113**, p. 621. Let us consider a function of the coordinates q_1, q_2, q_3 of an electron, that is equal to 1 when the point considered is situated in the interior of a certain volume V of space and equal to zero for every exterior point, and let us represent by $\psi(q, \alpha)$ and $\overline{\psi}(\alpha, q)$ the transformation functions that allow us to transform a physical quantity F, whose form is known as a matrix (q', q''), into a matrix (α', α''), $\alpha_1, \alpha_2, \alpha_3$ being the first integrals of the equation of motion. The function f, written as a matrix (q', q''), will then take the form $f(q')\delta(q' - q'')$, where $\delta(q' - q'')$ represents Dirac's unit matrix. As a matrix (α', α''), f will then take the form

$$\begin{aligned} f(\alpha', \alpha'') &= \int \bar{\psi}(\alpha', q')dq' f(q')\delta(q' - q'')dq''\psi(q'', \alpha'') \\ &= \int_V \bar{\psi}(\alpha', q')dq'\psi(q', \alpha'') \ , \end{aligned}$$

the integral having to be extended over the whole of the considered volume. The diagonal terms of $f(\alpha', \alpha'')$, which may be written in the

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[a] Again, these remarks do not appear in the published French and we have reproduced them from Bohr's *Collected Works* (Bohr 1985, p. 105) (*eds.*).

form

$$f(\alpha) = \int \psi\bar{\psi}dq \ ,$$

will directly represent, in accordance with Dirac's interpretation of the matrices, the probability that, for a state of the system characterised by given values of α, the coordinates of the electron are those of a point situated in the interior of V. As ψ is nothing other than the solution of Schrödinger's wave equation, we arrive at once at the interpretation of the expression $\psi\bar{\psi}$ under discussion.

MR HEISENBERG. — I do not agree with Mr Dirac when he says that, in the described experiment, nature makes a choice. Even if you place yourself very far away from your scattering material, and if you measure after a very long time, you are able to obtain interference by taking two mirrors. If nature had made a choice, it would be difficult to imagine how the interference is produced. Evidently, we say that this choice of nature can never be known before the decisive experiment has been done; for this reason, we can make no real objection to this choice, because the expression 'nature makes a choice' then implies no physical observation. I should rather say, as I did in my last paper, that the *observer himself* makes the choice,[a] because it is only at the moment where the observation is made that the 'choice' has become a physical reality and that the phase relationship in the waves, the power of interference, is destroyed.

MR LORENTZ. — There is then, it seems to me, a fundamental difference of opinion on the subject of the meaning of these choices made by nature.

To admit the possibility that nature makes a choice means, I think, that it is impossible for us to know in advance how phenomena will take place in the future. It is then indeterminism that you wish to erect as a principle. According to you there are events that we cannot predict, whereas until now we have always assumed the possibility of these predictions.

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[a] From Heisenberg's publication record, it is clear that he is here referring to his uncertainty paper, which had appeared in May 1927. There we find the statement that 'all perceiving is a choice from a plenitude of possibilities' (Heisenberg 1927, p. 197). When Heisenberg says, in his above comment on Dirac, that the observer 'makes' the choice, he seems to mean this in the sense of the observer *bringing about* the choice (*eds.*).

PHOTONS

MR KRAMERS. — During the discussion of Mr de Broglie's report, Mr Brillouin explained to us how radiation pressure is exerted in the case of interference and that one must assume an auxiliary stress. But how is radiation pressure exerted in the case where it is so weak that there is only one photon in the interference zone? And how does one obtain the auxiliary tensor in this case?

MR DE BROGLIE. — The proof of the existence of these stresses can be made only if one considers a cloud of photons.

MR KRAMERS. — And if there is only one photon, how can one account for the sudden change of momentum suffered by the reflecting object?

MR BRILLOUIN. — No theory currently gives the answer to Mr Kramers' question.

MR KRAMERS. — No doubt one would have to imagine a complicated mechanism, that cannot be derived from the electromagnetic theory of waves?

MR DE BROGLIE. — The dualist representation by corpuscles and associated waves does not constitute a definitive picture of the phenomena. It does not allow one to predict the pressures exerted on the different points of a mirror during the reflection of a single photon. It gives only the mean value of the pressure during the reflection of a cloud of photons.

MR KRAMERS. — What advantage do you see in giving a precise value to the velocity v of the photons?

MR DE BROGLIE. — This allows one to imagine the trajectory followed by the photons and to specify the meaning of these entities; one can thus consider the photon as a material point having a position and a velocity.

MR KRAMERS. — I do not very well see, for my part, the advantage that there is, for the description of experiments, in making a picture where the photons travel along well-defined trajectories.

Mr Einstein. — During reflection on a mirror, Mr L. de Broglie assumes that the photons move parallel to the mirror with a speed $c \sin \theta$; but what happens if the incidence is normal? Do the photons then have zero speed, as required by the formula ($\theta = 0$)?

Mr Piccard. — Yes. In the case of reflection, one must assume that the component of the velocity of the photons parallel to the mirror is constant. In the interference zone, the component normal to the mirror disappears. The more the incidence increases, the more the photons are slowed down. One thus indeed arrives at stationary photons in the limiting case of normal incidence.[a]

Mr Langevin. — In this way then, in the interference zone, the photons no longer have the speed of light; they do not then always have the speed c?

Mr de Broglie. — No, in my theory the speed of photons is equal to c only outside any interference zone, when the radiation propagates freely in the vacuum. As soon as there are interference phenomena, the speed of the photons becomes smaller than c.

Mr De Donder. — I should like to show how the research of Mr L. de Broglie is related to mine on some points.

By identifying the ten equations of the gravitational field and the four equations of the electromagnetic field with the fourteen equations of the wave mechanics of L. Rosenfeld, I have obtained[b] a *principle of correspondence* that clarifies and generalises that of O. Klein.[c]

In my principle of correspondence, there appear *the quantum current* and *the quantum tensor*. I will give the formulas for them later on; let it suffice to remark now that the example of correspondence that Mr de Broglie has expounded is in harmony with my principle.

Mr L. Rosenfeld[d] has given another example. Here, the mass is *conserved* and, moreover, one resorts to the quantum current. We add that

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[a] Note that here the wave train is tacitly assumed to be limited longitudinally. Cf. our discussion of the de Broglie-Pauli encounter, section 10.2 (*eds.*).

[b] *Bull. Ac. Roy. de Belgique*, *Cl. des Sc.* (5) **XIII**, ns. 8–9, session of 2 August 1927, 504–9. See esp. equations (5) and (8).

[c] *Zeitschr. f. Phys.* **41**, n. 617 (1927). See esp. equations (18), p. 414.

[d] L. Rosenfeld, 'L'univers à cinq dimensions et la mécanique ondulatoire (quatrième communication)', *Bull. Ac. Roy. Belg.*, *Cl. des Sc.*, October 1927. See esp. paragraphs 4 and 5.

this model of quantisation is also included, as a particular case, in our principle of correspondence.

Mr Lorentz has remarked, with some surprise, that the continuity equation for charge is preserved in Mr de Broglie's example. Thanks to our principle of correspondence, and to Rosenfeld's compatibility[27] theorem, one can show that it will always be so for the total current (including the quantum current) and for the theorem of energy and momentum. The four equations that express this last theorem are satisfied by virtue of the two generalised quantum equations of de Broglie-Schrödinger.

One further small remark, to end with. Mr de Broglie said that *relativistic* systems do not exist yet. I have given the theory of *continuous* or *holonomic* systems.[a] But Mr de Broglie gives another meaning to the word *system*; he has in mind *interacting* systems, such as the Bohr atom, the system of three bodies, etc. I have remarked recently[b] that the quantisation of these systems should be done by means of a $(ds)^2$ taken in a *configuration* space with $4n$ dimensions, n denoting the number of particles. In a paper not yet published, I have studied particular systems called *additive*.

MR LORENTZ. — The stresses of which you speak and which you call quantum, are they those of Maxwell?

MR DE DONDER. — Our quantum stresses must contain the Maxwell stresses as a particular case; this results from the fact that our principle of correspondence is derived (in part, at least) from Maxwell's equations, and from the fact that these quantum stresses here formally play the same role as the stresses of electrostriction[c] in Einsteinian Gravity. Let us recall, on this subject, that our principle of correspondence is also derived from the fundamental equations of Einsteinian Gravity. Mr de Broglie has, by means of his calculations, thus recovered the stresses of radiation.

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[a] *C. R. Acad. Sc. Paris*, 21 February 1927, and *Bull. Ac. Roy. Belgique*, *Cl. des Sc.*, 7 March 1927.

[b] *Bull. Ac. Roy. Belgique*, *Cl. des Sc.*, 2 August 1927. See esp. form. (22).

[c] For more details, see our Note: 'L'électrostriction déduite de la gravifique einsteinienne', *Bull. Ac. Roy. Belgique*, *Cl. des Sc.*, session of 9 October 1926, 673–8.

Photons and electrons

Mr Langevin makes a comparison between the old and modern statistics.

Formerly, one decomposed the phase space into cells, and one evaluated the number of representative points attributing an individuality to each constituent of the system.

It seems today that one must modify this method by suppressing the individuality of the constituents of the system, and substituting instead the individuality of the states of motion. By assuming that any number of constituents of the system can have the same state of motion, one obtains the statistics of Bose-Einstein.

One obtains a third statistics, that of Pauli-Fermi-Dirac,[28] by assuming that there can be only a single representative point in each cell of phase space.

The new type of representation seems more appropriate to the conception of photons and particles: since one attributes a complete identity of nature to them, it appears appropriate to not insist on their individuality, but to attribute an individuality to the states of motion.

In the report of Messrs Born and Heisenberg, I see that it results from quantum mechanics that the statistics of Bose-Einstein is suitable for molecules, that of Pauli-Dirac for electrons and protons. This means that for photons[29] and molecules there is superposition, while for protons and electrons there is impenetrability. Material particles are then distinguished from photons[30] by their impenetrability.[31]

Mr Heisenberg. — There is no reason, in quantum mechanics, to prefer one statistics to another. One may always use different statistics, which can be considered as complete solutions of the problem of quantum mechanics. In the current state of the theory, the question of interaction has nothing to do with the question of statistics.

We feel nevertheless that Einstein-Bose statistics could be the more suitable for light quanta, Fermi-Dirac statistics for positive and negative electrons.[a] The statistics could be connected with the difference between radiation and matter, as Mr Bohr has pointed out. But it is difficult to establish a link between this question and the problem of interaction. I shall simply mention the difficulty created by electron spin.

Mr Kramers reminds us of Dirac's research on statistics, which

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[a] That is, for protons and electrons (*eds.*).

has shown that Bose-Einstein statistics can be expressed in an entirely different manner. The statistics of photons, for example, is obtained by considering a cavity filled with blackbody radiation as a system having an infinity of degrees of freedom. If one quantises this system according to the rules of quantum mechanics and applies Boltzmann statistics, one arrives at Planck's formula, which is equivalent to Bose-Einstein statistics applied to photons.

Jordan has shown that a formal modification of Dirac's method allows one to arrive equally at a statistical distribution that is equivalent to Fermi statistics. This method is suggested by Pauli's exclusion principle.

MR DIRAC[a] points out that this modification, considered from a general point of view, is quite artificial. Fermi statistics is not established on exactly the same basis as Einstein-Bose statistics, since the natural method of quantisation for waves leads precisely to the latter statistics for the particles associated with the waves. To obtain Fermi statistics, Jordan had to use an unusual method of quantisation for waves, chosen specially so as to give the desired result. There are mathematical errors in the work of Jordan that have not yet been redressed.

MR KRAMERS. — I willingly grant that Jordan's treatment does not seem as natural as the manner by which Mr Dirac quantises the solution of the Schrödinger equation. However, we do not yet understand why nature requires this quantisation, and we can hope that one day we will find the deeper reason for why it is necessary to quantise in one way in one case and in another way in the other.

MR BORN. — An essential difference between Debye's old theory, in which the characteristic vibrations of the blackbody cavity are treated like Planck oscillators, and the new theory is this, that both yield quite exactly Planck's radiation formula (for the mean density of radiation), but that the old theory leads to inexact values for the local fluctuations of radiation, while the new theory gives these values exactly.

MR HEISENBERG. — According to the experiments, protons and electrons both have an angular momentum and obey the laws of the statistics of Fermi-Dirac; these two points seem to be related. If one takes two particles together, if one asks, for example, which statistics one

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[a] On this criticism by Dirac, cf. Kragh (1990, pp. 128–30) (*eds.*).

must apply to a gas made up of atoms of hydrogen, one finds that the statistics of Bose-Einstein is the right one, because by permuting two H atoms, we permute one positive electron and one negative electron,[a] so that we change the sign of the Schrödinger function *twice*. In other words, Bose-Einstein statistics is valid for all gases made up of neutral molecules, or more generally, composed of systems whose charge is an even multiple of e. If the charge of the system is an odd multiple of e, the statistics of Fermi-Dirac applies to a collection of these systems.

The He nucleus does not rotate and a collection of He nuclei obeys the laws of Bose-Einstein statistics.

MR FOWLER asks if the fine details of the structure of the bands of helium agree better with the idea that we have only symmetric states of rotation of the nuclei of helium than with the idea that we have only antisymmetric states.

MR HEISENBERG. — In the bands of helium, the fact that each second line disappears teaches us that the He nucleus is not endowed with a spinning motion. But it is not yet possible to decide experimentally, on the basis of these bands, if the statistics of Bose-Einstein or that of Fermi-Dirac must be applied to the nucleus of He.

MR SCHRÖDINGER. — You have spoken of experimental evidence in favour of the hypothesis that the proton is endowed with a spinning motion just like the electron, and that protons obey the statistical law of Fermi-Dirac. What evidence are you alluding to?

MR HEISENBERG. — The experimental evidence is provided by the work of Dennison[a] on the specific heat of the hydrogen molecule, work which is based on Hund's research concerning the band spectra of hydrogen.

Hund found good agreement between his theoretical scheme and the experimental work of Dilke, Hopfield and Richardson, by means of the hypotheses mentioned by Mr Schrödinger. But for the specific heat, he found a curve very different from the experimental curve. The experimental curve of the specific heat seemed rather to speak in favour of Bose-Einstein statistics. But the difficulty was elucidated in the paper by Dennison, who showed that the systems of 'symmetric' and 'anti-

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[a] That is, we permute the two protons, and also the two electrons (*eds.*).

[a] *Proc. Roy. Soc. A* **114** (1927), 483.

symmetric' terms (with regard to protons) do not combine in the time necessary to carry out the experiment. At low temperature, a transition takes place about every three months. The ratio of statistical weights of the systems of symmetric and antisymmetric terms is 1 : 3, as in the helium atom. But at low temperatures the specific heat must be calculated as if one had a mixture of two gases, an 'ortho' gas and a 'para' gas. If one wished to perform experiments on the specific heat with a gas of hydrogen, kept at low temperature for several months, the result would be totally different from the ordinary result.

Mr Ehrenfest wishes to formulate a question that has some relation to the recent experiments by Mr Langmuir on the disordered motion of electrons in the flow of electricity through a gas.

In the well-known Pauli exclusion (Pauliverbot), one introduces (at least in the language of the old quantum theory) a particular incompatibility relation between the quantum motions of the different particles of a single system, without speaking explicitly of the role possibly played by the forces acting between these particles. Now, suppose that through a small opening one allows particles that, so to speak, do not exert forces on each other, to pass from a large space into a small box bounded by quite rigid walls with a complicated shape, so that the particles encounter the opening and leave the box only at the end of a sufficiently long time. Before entering the box, if the particles have almost no motion relative to one another, the Pauli exclusion intervenes. After their exit, will they have very different energies, independently of the weakness of the mutual action between the particles? Or else what role do these forces play in the production of Pauli's incompatibility (choice of antisymmetric solutions of the wave equation)?

Mr Heisenberg. — The difficulty with Mr Ehrenfest's experiment is the following: the two electrons must have different energies. If the energy of interaction of the two electrons is very small, the time τ_1 required for the electrons to exchange an appreciable amount of energy is very long. But to find experimentally which state, symmetric or antisymmetric, the system of the two electrons in the box is in, we need a certain time τ_2 which is at least $\sim 1/\nu$, if $h\nu$ is the [energy] difference between the symmetric and antisymmetric states. Consequently, $\tau_1 \sim \tau_2$ and the difficulty disappears.

Mr Richardson. — The evidence for a nuclear spin is much more complete than Mr Heisenberg has just said. I have recently had occasion to classify a large number of lines in the visible bands of the spectrum of the H_2 molecule. One of the characteristic features of this spectrum is a rather pronounced alternation in the intensity of the successive lines. The intensities of the lines of this spectrum were recently measured by MacLennan, Grayson-Smith and Collins. Unfortunately, a large number of these lines overlap with each other, so that the intensity measurements must be accepted only with reservations.

But nevertheless, I think one can say, without fear of being mistaken, that all the bands that are sufficiently well-formed and sufficiently free of influences of the lines on each other (so that one can have confidence in the intensity measurements) have lines, generally numbered 1, 3, 5, ..., that are intrinsically three times more intense than the intermediate lines, generally numbered 2, 4, 6, ... By intrinsic intensity, I mean that which one obtains after having taken into account the effects on the intensity of temperature and quantum number (and also, of course, the effects of overlap with other lines, where it is possible to take this into account). In other words, I wish to say that the constant c of the intensity formula

$$\mathcal{J} = c\left(m + \frac{1}{2}\right) e^{\frac{-\left(m+\frac{1}{2}\right)^2 h^2}{8\pi K k T}} \ ,$$

where m is the number of the line and K the moment of inertia of the molecule, is three times bigger for the odd-numbered lines than for the even-numbered ones. This means that the ratio 3 : 1 applies, with an accuracy of about 5%, for at least five different vibration states of a three-electron state of excitation. It also applies to another state, which is probably 3^1P if the others are 3^3P. It is also shown, but in a less precise way, that it applies to two different vibration states of a state of excitation with four electrons.

At present, then, there is a great deal of experimental evidence that this nuclear spin persists through the different states of excitation of the hydrogen molecule.

Mr Langmuir. — The question has often been raised of a similarity in the relation between light waves and photons on the one hand, and de Broglie waves and electrons on the other. How far can this analogy be developed? There are many remarkable parallels, but also I should like to see examined if there are no fundamental differences between these relations. Thus, for example, an electron is characterised by a constant charge. Is there a constant property of the photon that may be compared with the charge of the electron? The speed of the electron is variable; is that of the photon also? The electromagnetic theory of light has suggested a multitude of experiments, which have added considerably to our knowledge. The wave theory of the electron explains the beautiful results of Davisson and Germer. Can one hope that this theory will be as fertile in experimental suggestions as the wave theory of light has been?[32]

MR EHRENFEST. — When one examines a system of plane waves of elliptically polarised light, placing oneself in differently moving coordinate systems, these waves show the same degree of ellipticity whatever system one places oneself in. Passing from the language of waves to that of photons, I should like to ask if one must attribute an elliptical polarisation (linear or circular in the limiting cases) to each photon? If the reply is affirmative, in view of the invariance of the degree of ellipticity in relativity, one must distinguish as many species of photons as there are degrees of ellipticity. That would yield, it seems to me, a new difference between the photon and the spinning electron. If, on the other hand, one wishes above all to retain the analogy with the electron, as far as I can see one comes up against two difficulties:

1. How then must one describe linearly polarised light in the language of photons? (It is instructive, in this respect, to consider the way in which the two linearly polarised components, emitted perpendicularly to the magnetic field by a flame showing the Zeeman effect, are absorbed by a second flame placed in a magnetic field with antiparallel orientation.)

Mr Zeeman, to whom I posed the question, was kind enough to perform the experiment about a year ago, and he was able to notice that the absorption is the same in parallel and antiparallel fields, as one could have predicted, in fact, by considerations of continuity.

2. For electrons, which move always with a speed less than that of light, the universality of the spin may be expressed as follows, that one transforms the corresponding antisymmetric tensor into a system of coordinates carried with the electron in its translational motion ('at rest'). But photons always move with the speed of light!

MR COMPTON. — Can light be elliptically polarised when the photon has an angular momentum?

Mr Ehrenfest. — Because the photons move with the speed of light, I do not really understand what it means when one says that each photon has a universal angular momentum just like an electron.

Allow me to remind you of yet another property of photons. When two photons move in directions that are not exactly the same, one can say quite arbitrarily that one of the photons is a radio-photon and the other a γ-ray photon, or inversely. That depends quite simply on the moving system of coordinates to which one refers the pair of photons.

Mr Lorentz. — Can you make them identical by such a transformation?

Mr Ehrenfest. — Perfectly. If they move in different directions. One can then give them the same colour by adopting a suitable frame of reference. It is only in the case where their worldlines are exactly parallel that the ratio of their frequencies remains invariant.

Mr Pauli. — The fact that the spinning electron can take two orientations in the field allowed by the quanta seems to invite us at first to compare it to the fact that there are, for a given direction of propagation of the light quanta, two characteristic vibrations of blackbody radiation, distinguished by their polarisation. Nevertheless there remain essential differences between the two cases. While in relativity one describes waves by a (real) sextuple vector $F_{ix} = -F_{xi}$, for the spinning electron one has proposed the following two modes of description for the associated de Broglie waves: 1. One describes these waves by two complex functions ψ_α, ψ_β (and so by four real functions); but these functions transform in a way that is hardly intuitive during the change from one system of coordinates to another. That is the route I followed myself. Or else: 2. Following the example of Darwin, one introduces a *quadruple* vector with generally complex components (and so eight real functions in total). But this procedure has the inconvenience that the vector involves a redundancy [indétermination], because all the verifiable results depend on only *two* complex functions.

These two modes of description are mathematically equivalent, but independently of whether one decides in favour of one or the other, it seems to me that one cannot speak of a *simple* analogy between the polarisation of light waves and the polarisation of de Broglie waves associated with the spinning electron.

Another essential difference between electrons and light quanta is this, that between light quanta there does not exist direct (immediate) mutual action, whereas electrons, as a result of their carrying an electric charge, exert direct mutual actions on each other.

MR DIRAC.[a] — I should like to point out an important failure in the analogy between the spin of electrons and the polarisation of photons. In the present theory of the spinning electron one assumes that one can specify the direction of the spin axis of an electron at the same time as its position, or at the same time as its momentum. Thus the spin variable of an electron commutes[33] with its coordinate and with its momentum variables. The case is different for photons. One can specify a direction of polarisation for plane monochromatic light waves, representing photons of given momentum, so that the polarisation variable commutes with the momentum variables. On the other hand, if the position of a photon is specified, it means one has an electromagnetic disturbance confined to a very small volume,[34] and one cannot give a definite polarisation, i.e. a definite direction for the electric vector, to this disturbance. Thus the polarisation variable of a photon does not commute with its coordinates.

MR LORENTZ. — In these different theories, one deals with the probability $\psi\psi^*$. I should like to see quite clearly how this probability can exist when particles move in a well-defined manner following certain laws. In the case of electrons, this leads to the question of motions in the field ψ (de Broglie). But the same question arises for light quanta. Do photons allow us to recover all the classical properties of waves? Can one represent the energy, momentum and Poynting vector by photons? One sees immediately that, when one has an energy density and energy flow, if one wishes to explain this by photons then the number of photons per unit volume gives the density, and the number of photons per second that move across a unit surface gives the Poynting vector.

The photons will then have to move with a speed different from that of light. If one wished to assign always the same speed c to the photons, in some cases one would have to assume a superposition of several photon currents. Or else one would have to assume that the photons cannot be used to represent all the components of the energy-momentum tensor. Some of the terms must be continuous in the field. Or else the photons are smeared out [fondus].

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[a] Again, here we follow Dirac's original English (*eds.*).

A related question is to know whether the photons can have a speed different from that of light and whether they can even be at rest. That would altogether displease me. Could we speak of these photons and of their motion in a field of radiation?

Mr de Broglie. — When I tried to relate the motion of the photons to the propagation of the waves ψ of the new mechanics, I did not worry about putting this point of view in accord with the electromagnetic conception of light waves, and I considered only waves ψ of scalar character, which one has normally used until now.

Mr Lorentz. — One will need these waves for photons also. Are they of a different nature than light waves? It would please me less to have to introduce two types of waves.

Mr de Broglie. — At present one does not know at all the physical nature of the ψ-wave of the photons. Can one try to identify it with the electromagnetic wave? That is a question that remains open. In any case, one can provisionally try to develop a theory of photons by associating them with waves ψ.

Mr Lorentz. — Is the speed of the wave equal to that of light?

Mr de Broglie. — In my theory, the speed of photons is equal to c, except in interfering fields. In general, I find that one must assign to a moving corpuscle a proper mass M_0 given by the formula

$$M_0 = \sqrt{m_0^2 - \frac{h^2}{4\pi^2 c^2} \frac{\Box a}{a}} \ ,$$

the function $\frac{\Box a}{a}$ being calculated at the point where the moving body is located at the given moment (a is the amplitude of the wave ψ). For photons, one has

$$m_0 = 0 \ .$$

Thus, when a photon moves freely, that is to say, is associated with an ordinary plane wave, M_0 is zero and, to have a finite energy, the photon must have speed c. But, when there is interference, $\frac{\Box a}{a}$ becomes different from zero, M_0 is no longer zero and the photon, to maintain the same energy, must have a speed less than c, a speed that can even be zero.

Mr Lorentz. — The term $\frac{\Box a}{a}$ must be negative, otherwise the mass would become imaginary.

Mr de Broglie. — In the corpuscular conception of light, the existence of diffraction phenomena occuring at the edge of a screen requires us to assume that, in this case, the trajectory of the photons is curved. The supporters of the emission theory said that the edge of the screen exerts a force on the corpuscle. Now, if in the new mechanics as I develop it, one writes the Lagrange equations for the photon, one sees appear on the right-hand side of these equations a term proportional to the gradient of M_0.

This term represents a sort of force of a new kind, which exists only when the proper mass varies, that is to say, where there is interference. It is this force that will curve the trajectory of the photon when its wave ψ is diffracted by the edge of a screen.

Furthermore, for a cloud of photons the same Lagrange equations lead one to recover the internal stresses pointed out by Messrs Schrödinger and De Donder.[a] One finds, indeed, the relations

$$\frac{\partial}{\partial x^k}\left[T^{ik} + \Pi^{ik}\right] = 0 \ ,$$

where the tensor T^{ik} is the energy-momentum tensor of the corpuscles

$$T^{ik} = \rho_0 u^i u^k \ .$$

The tensor Π^{ik}, which depends on derivatives of the amplitude of the wave ψ and is zero when this amplitude is constant, represents stresses existing in the cloud of corpuscles, and these stresses allow us to recover the value of the radiation pressure in the case of reflection of light by a mirror.

The tensor $T^{ik} + \Pi^{ik}$ is certainly related to the Maxwell tensor but, to see clearly how, one would have to be able to clarify the relationship existing between the wave ψ of the photons and the electromagnetic light wave.

Mr Pauli.[b] — It seems to me that, concerning the statistical results of scattering experiments, the conception of Mr de Broglie is in full agreement with Born's theory in the case of elastic collisions, but that it is no longer so when one also considers inelastic collisions. I should

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[a] Cf. Schrödinger (1927b) and De Donder's comments above (*eds.*).

[b] Cf. section 10.2 (*eds.*).

like to illustrate this by the example of the rotator, which was already mentioned by Mr de Broglie himself. As Fermi[a] has shown, the treatment by wave mechanics of the problem of the collision of a particle that moves in the (x, y) plane and of a rotator situated in the same plane, may be made clear in the following manner.[b] One introduces a configuration space of three dimensions, of which two coordinates correspond to the x and y of the colliding particle, while as third coordinate one chooses the angle φ of the rotator. In the case where there is no mutual action between the rotator and the particle, the function ψ of the total system is given by[35]

$$\psi(x, y, \varphi) = Ae^{2\pi i\left[\frac{1}{h}(p_x x + p_y y + p_\varphi \varphi) - \nu t\right]} ,$$

where one has put

$$p_\varphi = m\frac{h}{2\pi} \qquad (m = 0, 1, 2, ...) .$$

In particular, the sinusoidal oscillation of the coordinate φ corresponds to a stationary state of the rotator. According to Born, the superposition of several partial waves of this type, corresponding to different values of m and by consequence of p_φ,[36] means that there is a probability different from zero for several stationary states of the rotator, while according to the point of view of Mr de Broglie, in this case the rotator no longer has a constant angular velocity and can also execute oscillations in certain circumstances.

Now, in the case of a finite energy of interaction between the colliding particle and the rotator, if we study the phenomenon of the collision by means of the wave equation in the space (x, y, φ), according to Fermi the result can be interpreted very simply. Indeed, since the energy of interaction depends on the angle φ in a periodic manner and vanishes at large distances from the rotator, that is to say from the axis φ, in the space (x, y, φ) we are dealing simply with a wave that falls on a grating and, in particular, on a grating that is unlimited in the direction of the axis φ. At large distances from the grating, waves come out only in fixed directions in configuration space, characterised by integral values of the difference $m' - m''$. Fermi has shown that the different spectral orders correspond simply to the different possible ways of transferring the energy of the colliding particle to the rotator, or conversely. Thus to

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[a] *Zeitschr. f. Phys.* **40** (1926), 399.

[b] See section 10.2 for a discussion of Fermi's argument (*eds.*).

each spectral order of the grating corresponds a given stationary state of the rotator after the collision.

It is, however, an essential point that, in the case where the rotator is in a stationary state before the collision, the incident wave is unlimited in the direction of the axis. For this reason, the different spectral orders of the grating will always be superposed at each point of configuration space. If we then calculate, according to the precepts of Mr de Broglie, the angular velocity of the rotator after the collision, we must find that this velocity is not constant. If one had assumed that the incident wave is limited[a] in the direction of the axis φ, it would have been the same before the collision. Mr de Broglie's point of view does not then seem to me compatible with the requirement of the postulate of the quantum theory, that the rotator is in a stationary state both before and after the collision.

To me this difficulty does not appear at all fortuitous or inherent in the particular example of the rotator; in my opinion, it is due directly to the condition assumed by Mr de Broglie, that in the individual collision process the behaviour of the particles should be completely determined and may at the same time be described completely by ordinary kinematics in spacetime. In Born's theory, agreement with the quantum postulate is realised thus, that the different partial waves in configuration space, of which the general solution of the wave equation after the collision is composed, are applicable [indiquées] *separately* in a statistical way. But this is no longer possible in a theory that, in principle, considers it possible to avoid the application of notions of probability to *individual* collision processes.

Mr de Broglie. — Fermi's problem is not of the same type as that which I treated earlier; indeed, he makes configuration space play a part, and not ordinary space.

The difficulty pointed out by Mr Pauli has an analogue in classical optics. One can speak of the beam diffracted by a grating in a given direction only if the grating and the incident wave are laterally limited, because otherwise all the diffracted beams will overlap and be bathed in the incident wave. In Fermi's problem, one must also assume the wave ψ to be limited laterally in configuration space.

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[a] The French reads 'illimitée' [unlimited], which we interpret as a misprint. Pauli seems to be saying that if, on the other hand, the incident wave had been taken as limited, then *before* the collision the rotator could not have been in a stationary state and its angular velocity could not have been constant (*eds.*).

Mr Lorentz. — The question is to know what a particle should do when it is immersed in two waves at the same time.

Mr de Broglie. — The whole question is to know if one has the right to assume the wave ψ to be limited laterally in configuration space. If one has this right, the velocity of the representative point of the system will have a constant value, and will correspond to a stationary state of the rotator, as soon as the waves diffracted by the φ-axis will have separated from the incident beam.

One can say that it is not possible to assume the incident beam to be limited laterally, because Fermi's configuration space is formed by the superposition of *identical* layers of height 2π in the direction of the φ-axis; in other words, two points of configuration space lying on the same parallel to the φ-axis and separated by a whole multiple of 2π represent the *same* state of the system. In my opinion, this proves above all the artificial character of configuration spaces, and in particular of that which one obtains here by rolling out along a line the cyclic variable φ.

Mr De Donder. — In the course of the discussion of Mr L. de Broglie's report, we explained how we obtained our Principle of Correspondence; thanks to this principle, one will have[a]

$$\rho_{(e)}u^a + \Lambda^a = \sqrt{-g}K^2\frac{c}{e}\sum_n \frac{-h}{2i\pi}g^{an}\left(\psi\bar\psi_{.n} - \bar\psi\psi_{.n}\right) - 2\frac{e}{c}\Phi^a\psi\bar\psi \ ,$$

$$\rho_{(m)}u^a u^b + \Pi^{ab} = \sqrt{-g}\sum_\alpha\sum_\beta \gamma^{a\alpha}\gamma^{b\beta}\left(\psi_{.\alpha}\bar\psi_{.\beta} + \bar\psi_{.\alpha}\psi_{.\beta}\right) - \gamma^{ab}L$$

$$(a,\ b,\ n = 1, ..., 4;\ \alpha,\ \beta = 0, 1, ..., 4)\ .$$

The first relation represents *the total current* ($\equiv$ electronic current + quantum current) as a function of ψ and of the potentials g^{an}, Φ^a.

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[a] I adopt here L. Rosenfeld's notation, so as to facilitate the comparison with his formulas, given later.

Recall that one has set

$$L \equiv \sum_{\alpha} \sum_{\beta} \gamma^{\alpha\beta} \psi_{.\alpha} \overline{\psi_{.\beta}} + k^2 \left(\mu^2 - \frac{1}{2\chi} \right) \psi \overline{\psi} \ ,$$

$$\gamma^{ab} \equiv g^{ab} \ , \quad \gamma^{0a} \equiv -\alpha \Phi^a \ , \quad \gamma^{00} \equiv \alpha^2 \Phi^a \Phi_a - \frac{1}{\xi} \ ,$$

$$\xi \alpha^2 \equiv 2\chi \ , \quad \chi \equiv \frac{8\pi G}{c^2} \ , \quad G \equiv 6.7 \times 10^{-8} \text{ c.g.s.}$$

We have already mentioned the examples (or models) of correspondence found respectively by L. de Broglie and L. Rosenfeld. To be able to show clearly a new solution to the problem relating to *photons* that Mr L. de Broglie has just posed, I am going to display the formulas concerning the two above-mentioned models.[a]

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[a] L. Rosenfeld, 'L'univers à cinq dimensions et la mécanique ondulatoire', *Bull. Ac. Roy. Belgique*, *Cl. des Sc.*, October 1927. See respectively the formulas (*38´), (*31), (*27), (21), (1), (8), (35), (28), (29), (*35).

Model of L. de Broglie.	*Model of L. Rosenfeld.*
Quantum current $\Lambda_a \equiv 0$.	Quantum current $\Lambda_a = 2K^2A'^2C_{.a}$, where A' is the modulus of ψ and where the potential $C \equiv S' - S$. The function S satisfies the *classical* Jacobi equation; the function S' satisfies the *modified* Jacobi equation; one then has $$\gamma^{\alpha\beta}S_{.\alpha}S_{.\beta} = \mu^2 - \frac{1}{2\chi}\ ,$$ $$\gamma^{\alpha\beta}S'_{.\alpha}S'_{.\beta} = \mu^2 - \frac{1}{2\chi} + \frac{\Box A'}{K^2A'}\ .$$ The quantum potential C produces *the difference* between *physical* quantisation and *geometrical* quantisation. Recall that $\mu \equiv \frac{m_0c^2}{e}$, where m_0 and e are respectively the mass (at rest) and charge of the particle under consideration. We have also put $$k \equiv iK \equiv i\frac{2\pi}{h}\frac{e}{c}\ .$$
Charge density $\rho_{(e)} = 2K^2A'^2\mu'$, where we have put $$\mu'^2 = \mu^2 + \frac{\Box A'}{K^2A'}\ ,$$ which, retaining the charge e, reduces to substituting for the mass m_0 *the modified mass of L. de Broglie*: $$M_0 \equiv \sqrt{m_0^2 + \frac{h^2}{4\pi^2c^2}\frac{\Box A'}{A'}}\ .$$	Charge density $\rho_{(e)} = 2K^2A'^2\mu$. Here then *one retains*, at the same time, the mass m_0 and the charge e.

Let us respectively apply these formulas to the problem of the photon pointed out by Mr L. de Broglie. The *proper* mass m_0 of the photon is *zero*; in the model of Mr L. de Broglie, this mass must be replaced by the *modified* mass M_0; on the contrary, in the model of Mr L. Rosenfeld, one uses only the *proper* mass $m_0 \equiv 0$. In the two models, the charge density $\rho_{(e)}$ is zero. Finally, in the first model, the speed of the photon *must* vary; in contrast, in the second model, one can assume that this speed is always that of light. These conclusions obviously speak in favour of the model of L. Rosenfeld, and, in consequence, also in favour of the *physical* existence of our quantum current Λ^a $(a = 1, 2, 3, 4)$. This current will probably play a dominant role in still unexplained optical phenomena.[a]

MR LORENTZ. — Let us take an atom of hydrogen and let us form the Schrödinger function ψ.[37] We consider $\psi\psi^*$ as the probability for the presence of the electron in a volume element. Mr Born has mentioned all the trajectories in the classical theory: let us take them with all possible phases,[a] but let us now take the ψ corresponding to a single value W_n of energy and then let us form $\psi\psi^*$. Can one say that this product $\psi_n\psi_n^*$ represents the probability that the electrons move with the given energy W_n? We think that the electron cannot escape from a certain sphere. The atom is limited, whereas ψ extends to infinity. That is disagreeable.[38]

MR BORN. — The idea that $\psi\psi^*$ represents a probability density has great importance in applications. If, for example, in the classical theory an electron had two equilibrium positions separated by a considerable potential energy, then classically, for a sufficiently weak total energy only one oscillation could ever take place, around one of the two equilibrium positions. But according to quantum mechanics, each eigenfunction extends from one domain into the other; for this reason there always exists a probability that a particle, which at first vibrates in the neighbourhood of one of the equilibrium positions, jumps to the other. Hund has made important applications of this to molecular structure. This phenomenon probably also plays a role in the explanation of metallic conduction.

MR DE BROGLIE. — In the old theory of the motion of an electron in the hydrogen atom, an electron of total energy

$$W = \frac{m_0 c^2}{\sqrt{1-\beta^2}} - \frac{e^2}{r}$$

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[a] On this subject, Mr L. Brillouin has kindly drawn my attention to the experiments by Mr F. Wolfers: 'Sur un nouveau phénomène en optique: interférences par diffusion' (*Le Journal de Physique et le Radium* (VI) **6**, n. 11, November 1925, 354–68).

[a] The 'phases' of classical trajectories seems to be meant in the sense of action-angle variables (*eds.*).

cannot escape from a sphere of radius

$$R = -\frac{e^2}{W - m_0c^2}$$

because the value of the term $\frac{m_0c^2}{\sqrt{1-\beta^2}}$ has m_0c^2 as a lower limit.

In my conception one must take

$$W = \frac{M_0c^2}{\sqrt{1-\beta^2}} - \frac{e^2}{r} \ ,$$

as the expression for the energy, where M_0 is the variable proper mass which I have already defined. Calculation shows that the proper mass M_0 diminishes when r increases, in such a way that an electron of energy W is no longer at all constrained to be in the interior of a sphere of radius R.

Mr Born. — Contrary to Mr Schrödinger's opinion, that it is nonsense to speak of the location and motion of an electron in the atom, Mr Bohr and I are of the opinion that this manner of speaking always has a meaning when one can specify an experiment allowing us to measure the coordinates and the velocities with a certain approximation.

Again in Richardson's notes on the general discussion (cf. p. 478), the following text together with Fig. D (both labelled 'Bohr'), and a similar figure with the shaded region labelled 'B', appear immediately after notes on De Donder's lengthy exposition just above, and clearly refer to remarks Bohr made on the topic being addressed here:

B[ohr] says it has no point to worry about the paradox that the electron in the atom is in a fixed path (ellipse or circle) and the probability that it should be found in a given place is given by the product $\psi\bar{\psi}$ which is a continuous function of space extending from zero to ∞. He says if we take a region such as B a long way from the atom in order to find if the electron is there we must illuminate it with long light waves and the frequency of these is so low that the electron is out of the region by reason of its motion in the stationary state before it has been illuminated long enough for the photoelectric act to occur. I am really not sure if this is right. But, anyway, it is no objection to pulling it out with an intense *static* electric field & this appears to be what is happening in the W experiments.

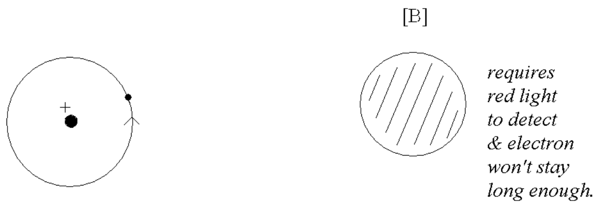

Fig. D.

Mr Pauli. — One can indeed determine the location of the electron outside the sphere, but without modifying its energy to the point where an ionisation of the atom occurs.

Mr Lorentz. — I should like to make a remark on the subject of wave packets.[a]

When Mr Schrödinger drew attention to the analogy between mechanics and optics, he suggested the idea of passing from corpuscular mechanics to wave mechanics by making a modification analogous to that which is made in the passage from geometrical optics to wave optics.[39] The wave packet gave a quite striking picture of the electron, but in the atom the electron had to be completely smeared out [fondu], the packet having the dimensions of the atom. When the dimensions of the wave packet become comparable to those of the trajectories of the classical theory, the material point would start to spread; having passed this stage, the electron will be completely smeared out.

The mathematical difficulty of constructing wave packets in the atom is due to the fact that we do not have at our disposal wavelengths sufficiently small or sufficiently close together. The frequencies of stable waves in the atom (eigenvalues) are more or less separated from each other; one cannot have frequencies very close together corresponding to states differing by very little, because the conditions at infinity would not

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[a] Cf. also the discussion of the Lorentz-Schrödinger correspondence in section 4.3 (*eds.*).

be satisfied. To construct a packet, one must superpose waves of slightly different wavelengths; now, one can use only eigenfunctions ψ_n, which are sharply different from each other. In atoms, then, one cannot have wave packets. But there is a difficulty also for free electrons, because in reality a wave packet does not, in general, retain its shape in a lasting manner. Localised [limités] wave packets do not seem able to maintain themselves; spreading takes place. The picture of the electron given by a wave packet is therefore not satisfying, except perhaps during a short enough time.

What Mr Bohr does is this: after an observation he again localises [limite] the wave packet so as to make it represent what this observation has told us about the position and motion of the electron; a new period then starts during which the packet spreads again, until the moment when a new observation allows us to carry out the reduction again. But I should like to have a picture of all that during an unlimited time.[40]

Mr Schrödinger. — I see no difficulty at all in the fact that on orbits of small quantum number one certainly cannot construct wave packets that move in the manner of the point electrons of the old mechanics.

The fact that this is impossible is precisely the salient point of the wave mechanical view, the basis of the absolute powerlessness of the old mechanics in the domain of atomic dimensions. The original picture was this, that what moves is in reality not a point but a domain of excitation of finite dimensions, in particular at least of the order of magnitude of a few wavelengths. When such a domain of excitation propagates along a trajectory whose dimensions and radii of curvature are large compared with the dimensions of the domain itself, one can abstract away the details of its structure and consider only its progress along the trajectory. This progress takes place following exactly the laws of the old mechanics. But if the trajectory shrinks until it becomes of the order of magnitude of a few wavelengths, as is the case for orbits of small quantum number, all its points will be continually inside the domain of excitation and one can no longer reasonably speak of the propagation of an excitation along a trajectory, which implies that the old mechanics loses all meaning.

That is the original idea. One has since found that the naive identification of an electron, moving on a macroscopic orbit, with a wave packet encounters difficulties and so cannot be accepted to the letter. The main difficulty is this, that with certainty the wave packet spreads in all directions when it strikes an obstacle, an atom for example. We know today, from the interference experiments with cathode rays by

Davisson and Germer, that this is part of the truth, while on the other hand the Wilson cloud chamber experiments have shown that there must be something that continues to describe a well-defined trajectory after the collision with the obstacle. I regard the compromise proposed from different sides, which consists of assuming a combination of waves and point electrons, as simply a provisional manner of resolving the difficulty.

MR BORN. — Also in the classical theory, the precision with which the future location of a particle can be predicted depends on the accuracy of the measurement of the initial location. It is then not in this that the manner of description of quantum mechanics, by wave packets, is different from classical mechanics. It is different because the laws of propagation of packets are slightly different in the two cases.

Notes to the translation

1 Microfilmed in AHQP-BMSS-11, section 5.
2 AHQP-36, section 10.
3 These notes are to be found in the Richardson collection in Houston, included with the copy of Born and Heisenberg's report (microfilmed in AHQP-RDN, document M-0309).
4 Included in AHQP-RDN, document M-0309.
5 French edition: 'les' is misprinted as 'le'.
6 AEA 16-617.00 (in German, with transcription and archival comments).
7 The French text has 'ψ' instead of 'ϕ', and 'doit satisfaire dans l'espace ordinaire' instead of the other way round. Note that ϕ is a functional of 'material' waves which themselves propagate in ordinary space.
8 AHQP-36, section 10.
9 The French adds: 'décrite par M. Compton'.
10 The French reads: 'provient uniquement de'.
11 Dirac's manuscript omits 'At present'.
12 The French reads 'intuitivité'.
13 Instead of 'commute' the French has 'permuter leurs valeurs'.
14 The French reads: 'transformation canonique'.
15 Instead of 'commute' the French has 'changent de valeur'.
16 Dirac's manuscript reads 'the'.
17 In Dirac's manuscript, the words 'determinism and' are cancelled and possibly reinstated. They appear in the French, which also omits 'of yesterday'.
18 <therefore unsatisfactory> <untenable>, the latter seems reinstated. The French has 'indéfendable'.
19 Instead of 'the distinguishing characteristics' the French has 'l'essence physique'.
20 <do>, {would} appears above the line, {can} below. The French reads 'ne figurent pas'.
21 This sentence does not appear in the French.
22 In the French, this sentence appears at the beginning of the paragraph.
23 This sentence does not appear in the French.
24 Dirac's manuscript has 'c_n^2'.
25 <Thus <a possibility> {the existence} of> .
26 The French has 'F.'.
27 Misprinted as 'comptabilité', despite having been corrected in the galley proofs.
28 In the printed text, the word 'Dirac' is misplaced to later in the paragraph.
29 Misprinted as 'protons'.
30 Again misprinted as 'protons'.
31 The version of this contribution in the galley proofs reads as follows:

Mr Langevin makes a comparison between the old and modern statistics.

Formerly, one decomposed the phase space, into cells and one evaluated the representative points.

It seems that one must modify this method by suppressing the individuality of the representative points and [blank]

Third method: that of Pauli.

This type of representation seems more appropriate to the conception of photons and particles [blank] attribute identity of nature, attribute at the same time individuality representing a state.

In the report of Messrs Born and Heisenberg, I see that it results from quantum mechanics that the statistics of Bose-Einstein is suitable for molecules, that of Pauli-Dirac, instead, is suitable for electrons. This means that for [blank] there is superposition, while for photons and electrons there is impenetrability.

32 The galley proofs contain the following version of this contribution:

Mr Langmuir would like to see established clearly a parallel between electrons and photons. What characterises an electron? A well-defined charge. What characterises the photon? Its velocity, perhaps? What is the analogy, what are the differences? Electron: de Broglie waves; photon: electromagnetic waves. For certain respects, this parallelism is clear, but perhaps it can be pursued to the end? What are the suggestions in the way of experiments?

33 The French renders 'commute' throughout with 'changer'.
34 The French adds: 'à un instant donné'.
35 'h' misprinted as 'λ'.
36 'p_φ' misprinted as 'φ'.
37 'ψ' missing in the original, with a space instead.
38 Here the galley proofs include an additional sentence:

If one took the integral extended over the whole of this space, the exterior part would be comparable.

39 The original mistakenly reads 'geometrical mechanics' and 'corpuscular optics'.
40 The version in the galley proofs reads as follows. (Note that in the case of this and the preceding contribution by Lorentz in the galley proofs, the published version was clearly not edited by him, since he had died at the beginning of February.)

Mr Lorentz. — I should like to make a remark on the subject of wave packets.

When Mr Schrödinger drew attention to the analogy between mechanics and optics, he suggested the idea of passing from geometrical mechanics to wave mechanics by making a modification analogous to that which is made in the passage from corpuscular optics to wave optics. The wave packet was a quite striking picture, but in the atom the electron is completely smeared out, the packet being of the dimensions of the atom [blank], material point that would start to spread [blank], passed, these electrons are completely smeared out.

Mathematical difficulty, wave packets in the atom, more or less distinguished frequencies (eigenvalues), but you could not have frequencies very close together by states differing by much or little [par des états tant soit peu différants], because one would not have the conditions at infinity. To construct a packet, one must superpose waves of slightly different wavelengths; now, one can use only eigenfunctions ψ_n, which are sharply different from each other. Thus one does not have the waves with which one could build a packet. In atoms, then, one cannot have the wave packets; it is the same for free electrons. All these wave packets will end up dissolving.

In reality a wave packet does not last; wave packets that would remain localised [limités] do not seem to maintain themselves; spreading takes place; the picture is therefore not satisfying [blank], short enough time perhaps [blank].

What Mr Bohr does is this [small blank] after an observation we have again localised [limité] [blank]; a new period starts [blank]. But I should like to have a picture of all that during an indefinite time.

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