Similarly $\quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$

and $[a_i, a_j^\dagger] = 0$ for $i \neq j \Rightarrow [a_i, a_j^\dagger] = \delta_{ij}$

The entire Fock space can now be constructed by continued application of creation operators:

- $|i\rangle = a_i^\dagger |0\rangle$ single particle states
- $(a_i^\dagger)^2 |0\rangle = a_i^\dagger | .. n_i = 1 .. \rangle = \sqrt{2}\, | .. n_i = 2 .. \rangle$
- $a_i^\dagger a_j^\dagger |0\rangle = | .. n_i = 1 .. n_j = 1 .. \rangle$ for $i \neq j$

$$\Rightarrow \quad |n_1 n_2 .. n_M\rangle = \prod_{k=1}^{M} \frac{1}{\sqrt{n_k!}} (a_k^\dagger)^{n_k} |0\rangle$$

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b) Fock space for fermions

The general antisymmetrized basis state for N particles is

$$|n_1 .. n_M\rangle = S_- |i_1 .. i_N\rangle = \frac{1}{\sqrt{N!}} \sum_P (-1)^P P |i_1 .. i_N\rangle$$

$$= \frac{1}{\sqrt{N!}} \det \begin{pmatrix} |i_1\rangle_1 & |i_1\rangle_2 & \cdots & |i_1\rangle_N \\ |i_2\rangle_1 & & & \vdots \\ \vdots & & & \vdots \\ |i_N\rangle_1 & \cdots & \cdots & |i_N\rangle_N \end{pmatrix}$$ Slater determinant

Since each state can be occupied by at most one particle, we have $M \geq N$ and the occupation numbers are

$$n_i = 0, 1$$

The Fock space is defined analogously to the bosonic case:

$$\left.\begin{aligned} \mathcal{H}^{(A)}_{\text{Fock}} &= \bigoplus_{N=0}^{\infty} \mathcal{H}^{(A)}_N \\ \mathcal{H}^{(A)}_0 &= \text{Span}\{ |0 \ldots 0\rangle \} \end{aligned}\right\}$$

The occupation number states are again orthonormal and complete:

$$\left.\begin{aligned} \langle n_1' n_2' \ldots | n_1 n_2 \ldots \rangle &= \delta_{n_1 n_1'} \delta_{n_2 n_2'} \ldots \\ \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \ldots | n_1 n_2 \ldots \rangle \langle n_1 n_2 \ldots | &= \mathbb{1} \end{aligned}\right\}$$

To construct the fermionic creation operators we define

$$S_- | i_1 \ldots i_N \rangle = a^+_{i_1} a^+_{i_2} \ldots a^+_{i_N} |0\rangle$$

Antisymmetry implies

$$S_- | \underline{i_1 i_2} i_3 \ldots i_N \rangle = - S_- | \underline{i_2 i_1} i_3 \ldots i_N \rangle$$

$$\Rightarrow \underline{a^+_{i_1} a^+_{i_2}} a^+_{i_3} \ldots |0\rangle = \underline{- a^+_{i_2} a^+_{i_1}} a^+_{i_3} \ldots |0\rangle$$

$\Rightarrow$ fermionic creation operators <u>anticommute</u>:

$$\{ a^+_i, a^+_j \} = a^+_i a^+_j + a^+_j a^+_i = 0$$

This implements in particular the Pauli principle

$$(a_i^+)^2 = 0$$

An occupation number state with $n_i = 0, 1$ is thus given by

$$|n_1 n_2 \ldots\rangle = (a_1^+)^{n_1} (a_2^+)^{n_2} \ldots |0\rangle$$

Application of a_i^+ to this state gives

$$a_i^+ |n_1 n_2 \ldots\rangle = \underbrace{a_i^+ (a_1^+)^{n_1} (a_2^+)^{n_2}}_{\text{exchange } \sum_{j<i} n_j \text{ times}} \ldots |0\rangle =$$

$$= (-1)^{\sum_{j<i} n_j} (a_1^+)^{n_1} \ldots (a_i^+)^{n_i+1} (a_{i+1}^+)^{n_{i+1}} \ldots |0\rangle =$$

$$= \begin{cases} 0 & n_i = 1 \\ (-1)^{\sum_{j<i} n_j} |n_1 \ldots n_i+1 \ldots\rangle & n_i = 0 \end{cases}$$

$$\Rightarrow a_i^+ |n_1 n_2 \ldots\rangle = (1-n_i)(-1)^{\sum_{j<i} n_j} |n_1 \ldots n_i+1 \ldots\rangle$$

The adjoint relation reads

$$\langle \ldots n_i \ldots | a_i = \langle \ldots n_i+1 \ldots | (1-n_i)(-1)^{\sum_{j<i} n_j}$$

$$\Rightarrow \langle \ldots n_i \ldots | a_i | \ldots n_i' \ldots \rangle = (1-n_i)(-1)^{\sum_{j<i} n_j} \underbrace{\delta_{n_i', n_i+1}}_{n_i = n_i' - 1}$$

Now $(1-n_i)\,\delta_{n_i', n_i+1} = (2-n_i')\,\delta_{n_i', n_i+1} =$

$$= \left\{ \begin{matrix} 0 & n_i' = 0 \\ 1 & n_i' = 1,\ n_i = 0 \end{matrix} \right\} = n_i'$$

$$\Rightarrow \quad a_i | \ldots n_i \ldots \rangle = n_i (-1)^{\sum_{j<i} n_j} | \ldots n_i - 1 \ldots \rangle$$

for $n_i = 1$, and $a_i | \ldots n_i = 0 \ldots \rangle = 0$

It follows that

$$a_i^\dagger a_i | \ldots n_i \ldots \rangle = a_i^\dagger n_i (-1)^{\sum_{j<i} n_j} | \ldots n_i - 1 \ldots \rangle$$

$$= \underbrace{(1-(n_i-1))\, n_i}_{} \underbrace{\left((-1)^{\sum_{j<i} n_j}\right)^2}_{=1} | \ldots n_i \ldots \rangle$$

$$= (2-n_i)\, n_i =$$

$$= \left\{ \begin{matrix} 0 & n_i = 0 \\ 1 & n_i = 1 \end{matrix} \right\} = n_i$$

$\Rightarrow$ $N_i := a_i^\dagger a_i$ acts again as a particle number operator:

$$N_i | \ldots n_i \ldots \rangle = n_i | \ldots n_i \ldots \rangle$$

On the other hand

$$a_i a_i^\dagger | \ldots n_i \ldots \rangle = (1-n_i)(n_i+1) | \ldots n_i \ldots \rangle =$$

$$= (1-n_i^2)\,|\ldots n_i \ldots\rangle = (1-n_i)\,|\ldots n_i \ldots\rangle$$

$$\Rightarrow (a_i^\dagger a_i + a_i a_i^\dagger)\,|\ldots n_i \ldots\rangle = |\ldots n_i \ldots\rangle$$

$$\Rightarrow \quad \{a_i, a_i^\dagger\} = \mathbb{1}$$

The complete fermionic anticommutation relations are

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$$

$$\{a_i, a_i^\dagger\} = \mathbb{1}$$

3° Representation of operators in second quantization

a) Bosons

Consider a operator T that can be written as a sum of single-particle operators

$$T = \sum_{\alpha=1}^{N} t_\alpha \qquad t_\alpha \text{ acts a particle } \alpha$$

Example: Kinetic energy, $t_\alpha = \frac{1}{2m}\vec{p}_\alpha^{\,2}$

In the basis of single particle states we can write $(t = \mathbb{1}\, t\, \mathbb{1})$

$$t = \sum_{i,j} |i\rangle \underbrace{\langle i | t | j\rangle}_{t_{ij} \text{ matrix element}} \langle j| = \sum_{i,j} |i\rangle t_{ij} \langle j|$$

$$\Rightarrow \quad T = \sum_{ij} t_{ij} \underbrace{\sum_{\alpha=1}^{N} |i\rangle_\alpha \langle j|_\alpha}_{=:\ \sigma_{ij}}$$

Action of σ_{ij} on Fock space states:

$$\sigma_{ij} |n_1 n_2 \ldots\rangle = \sum_\alpha |i_\alpha\rangle\langle j|_\alpha |n_1 \ldots n_i \ldots n_j \ldots\rangle$$

$$= \sum_\alpha |i\rangle_\alpha \langle j|_\alpha \frac{1}{\prod_k \sqrt{n_k!}} S_+ |i_1 \ldots i_N\rangle =$$

[Particles are identical $\Rightarrow [\sigma_{ij}, S_+] = 0$]

$$= \frac{1}{\prod_k \sqrt{n_k!}} S_+ \sum_\alpha |i\rangle_\alpha \underbrace{\langle j|_\alpha |i_1 \ldots i_N\rangle}$$

$$= \langle j|_\alpha |i_1\rangle |i_2\rangle \ldots |i_\alpha\rangle \ldots |i_N\rangle =$$

$$= \begin{cases} |i_1\rangle \ldots |i_{\alpha-1}\rangle |i_{\alpha+1}\rangle \ldots |i_N\rangle & j = i_\alpha \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \sigma_{ij} |n_1 \ldots n_i \ldots n_j \ldots\rangle =$$

$$= \frac{1}{\prod_k \sqrt{n_k!}} S_+ \sum_\alpha |i_1\rangle \ldots \overset{\alpha}{|i\rangle} \ldots |i_N\rangle \, \delta_{j, i_\alpha}$$

Effectively a particle has been moved from state j to state i.

After symmetrization, the state is thus proportional to $|n_1 \ldots n_i+1, n_j-1, \ldots\rangle$, but with a change in normalization:

$$\underline{\sigma_{ij}\,|n_1 \ldots n_i \ldots n_j \ldots\rangle} =$$

$$= n_j \sqrt{\frac{n_i+1}{n_j}}\;|n_1 \ldots, n_i+1, \ldots, n_j-1, \ldots\rangle$$

number of particles with $i_\alpha = j$

$$= \sqrt{n_j}\,\sqrt{n_i+1}\;|n_1 \ldots, n_i+1, \ldots, n_j-1, \ldots\rangle =$$

$$= \underline{a_i^+ a_j\,|n_1 \ldots n_i \ldots n_j \ldots\rangle}$$

This holds also for $i=j$:

$$\left.\begin{aligned}\sigma_{ii}\,|n_1 \ldots n_i \ldots\rangle &= n_i\,|n_1 \ldots n_i \ldots\rangle = \\ &= a_i^+ a_i\,|n_1 \ldots n_i \ldots\rangle\end{aligned}\right\}$$

Thus the representation of T in second quantization is

$$\underline{T = \sum_{ij} t_{ij}\, a_i^+ a_j}$$