A general two-particle operator takes the form

$$F = \frac{1}{2} \sum_{\alpha \neq \beta} \sum_{\substack{i,j,\\ k,l}} |i\rangle_\alpha |j\rangle_\beta \underbrace{\langle i,j | f | k,l \rangle}_{\text{two-particle matrix element}} \langle k|_\alpha \langle l|_\beta$$

<u>Example:</u> Pair interaction $v^{(2)}(\vec{r}, \vec{r}\,')$

$$\Rightarrow \langle i,j | v^{(2)} | k,l \rangle =$$

$$= \int d\vec{r} \int d\vec{r}\,' \, \psi_i^*(\vec{r}) \, \psi_j^*(\vec{r}\,') \, v^{(2)}(\vec{r} - \vec{r}\,') \, \psi_k(\vec{r}) \, \psi_l(\vec{r}\,')$$

Here the operator of interest is

$$\sigma_{ijkl} := \sum_{\alpha \neq \beta} |i\rangle_\alpha |j\rangle_\beta \langle k|_\alpha \langle l|_\beta =$$

$$= \underbrace{\left(\sum_\alpha |i\rangle_\alpha \langle k|_\alpha \right)}_{= a_i^\dagger a_k} \underbrace{\left(\sum_\beta |j\rangle_\beta \langle l|_\beta \right)}_{a_j^\dagger a_l} -$$

$$- \sum_\alpha |i\rangle_\alpha \underbrace{\langle k|_\alpha |j\rangle_\alpha}_{= \delta_{kj}} \langle l|_\alpha =$$

$$= a_i^\dagger \underbrace{a_k a_j^\dagger}_{= a_j^\dagger a_k + \delta_{kj} \mathbb{1}} a_l - \delta_{kj} a_i^\dagger a_l = a_i^\dagger a_j^\dagger a_k a_l$$

Thus σ_{ijkl} replaces two particles in states k,l by two particles in states i,j, and we can write

$$\underline{F = \frac{1}{2} \sum_{ijkl} \langle i,j | f | k,l \rangle a_i^+ a_j^+ a_k a_l}$$

b) Fermions

We will show that the relation

$$\sigma_{ij} = \sum_{\alpha=1}^{N} |i\rangle_\alpha \langle j|_\alpha = a_i^+ a_j$$

holds also for fermions. For this, consider an antisymmetrized state $S_- |i_1 \ldots, i_N\rangle$ with the ordering convention that

$$i_1 < i_2 < \ldots < i_N$$

Then $[\sigma_{ij}, S_-] = 0$

$$\sigma_{ij} S_- |i_1 \ldots i_N\rangle \overset{\downarrow}{=} S_- \sum_\alpha \underbrace{|i\rangle_\alpha \langle j|_\alpha |i_1 \ldots, i_N\rangle}$$

$$= \begin{cases} |i_1 \ldots i \ldots i_N\rangle & \text{if } i_\alpha = j \text{ and } i_\beta \neq i \text{ for } \beta \neq \alpha \\ 0 & \text{else} \end{cases}$$

Two cases:

(i) $i = j$: The state is reproduced if $n_i = 1$

$$\Rightarrow \sigma_{ii} S_- |i_1 \ldots i_N\rangle = n_i S_- |i_1 \ldots i_N\rangle =$$

$= n_i |n_1 n_2 \dots\rangle = a_i^\dagger a_i |n_1 n_2 \dots\rangle$

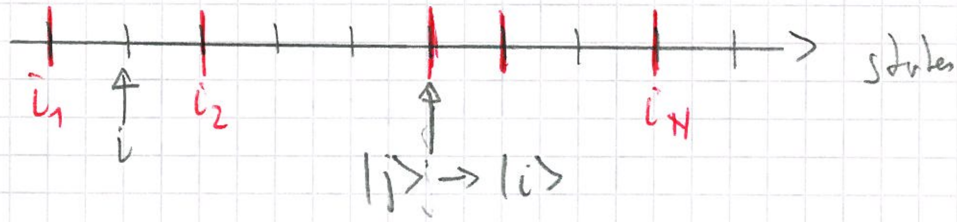
(ii) $i \neq j$: Then the result is nonzero only if $n_j = 1$ and $n_i = 0$

$\Rightarrow \sigma_{ij} S_- |i_1 \dots i_N\rangle = n_j (1-n_i) S_- |i_1 \dots i_N\rangle_{j \to i}$

where $|i_1 \dots i_N\rangle_{j\to i}$ is the product state with $|j\rangle$ replaced by $|i\rangle$.

To proceed this state has to be reordered. Again there are two cases:

(i) $i < j$:



To reorder the state, the number of exchanges needed is

$$K(i,j) = \sum_{i<k<j} n_k$$

$\Rightarrow \sigma_{ij} |n_1 \dots n_i \dots n_j\rangle = n_j (1-n_i)(-1)^{K(i,j)} |\dots n_i+1 \dots n_j-1 \dots\rangle$

Compare this to

$a_i^\dagger a_j |n_1 \dots n_i \dots n_j\rangle = a_i^\dagger n_j (-1)^{\sum_{k<j} n_k} |\dots n_i \dots n_j-1 \dots\rangle$

$= (1-n_i) n_j (-1)^{\sum_{k<j} n_k} (-1)^{\sum_{k<i} n_k} |\dots n_i+1 \dots n_j-1 \dots\rangle$

Now $\sum_{k<j} n_k + \sum_{k<i} n_k = 2 \sum_{k<i} n_k + K(i,j) + \underbrace{n_i}_{=0}$

□

(ii) $i > j$: The number of exchanges is again

$$K(i,j) = \sum_{j<k<i} n_k$$

but now

$$a_i^+ a_j |n_1 .. n_j .. n_i .. \rangle = a_i^+ n_j (-1)^{\sum_{k<j} n_k} | .. n_j - 1 .. \rangle$$

$$= (1-n_i) n_j \underbrace{(-1)^{\sum_{k<i} n_k - 1} (-1)^{\sum_{k<j} n_k}} | .. n_j - 1 .. n_i + 1 .. \rangle$$

total exponent: $\sum_{k<i} n_k + \sum_{k<j} n_k - 1 =$

$$= 2 \sum_{k<j} n_k + K(i,j) + \underbrace{n_j - 1}_{=0} \qquad \square$$

This proves the representation

$$T = \sum_{i,j} t_{ij} a_i^+ a_j$$

for single particle operators.

For two-particle operators consider

$$\sigma_{ijkl} = \sum_{\alpha \neq \beta} |i\rangle_\alpha |j\rangle_\beta \langle k|_\alpha \langle l|_\beta =$$

$$= a_i^+ \underbrace{a_k a_j^+}_{= \delta_{kj} - a_j^+ a_k} a_l - \delta_{kj} a_i^+ a_l = a_i^+ a_j^+ a_l a_k$$

To summarize, a general Hamiltonian including an external potential and pair interactions can be written as

$$H = \sum_{ij} \left(t_{ij} + V_{ij}^{(1)} \right) a_i^\dagger a_j +$$

$$+ \frac{1}{2} \sum_{ijkl} \langle ij | V^{(2)} | kl \rangle \, a_i^\dagger a_j^\dagger a_l a_k$$

The difference between bosons and fermions enters only through the commutation relations for the annihilation and creation operators.

4° Field operators

Creation and annihilation operators are always defined w.r.t. a single particle basis. Under a change of basis, $|i\rangle \to |\lambda\rangle$ the basis states transform as

$$|\lambda\rangle = \sum_i |i\rangle\langle i|\lambda\rangle$$

The corresponding operators satisfy

$$a_i^+ |0\rangle = |i\rangle , \quad a_\lambda^+ |0\rangle = |\lambda\rangle$$

$$\Rightarrow \quad a_\lambda^+ |0\rangle = \sum_i \langle i|\lambda\rangle a_i^+ |0\rangle$$

$$\Rightarrow \left\{ \begin{aligned} a_\lambda^+ &= \sum_i \langle i|\lambda\rangle a_i^+ \\ a_\lambda &= \sum_i \langle \lambda|i\rangle a_\lambda \end{aligned} \right.$$

In this section we formulate the second quantized theory w.r.t. the position and momentum representation.

---

a) Position representation

Single particle states are represented by wave function

$$\varphi_i(\vec{r}) = \langle \vec{r}|i\rangle , \quad \varphi_i^* = \langle i|\vec{r}\rangle$$

The corresponding creation and annihilation operators are called field operators and defined by

$$\left.\begin{aligned} \Psi^{+}(\vec{r}) &= \sum_i \varphi_i^{*}(\vec{r})\, a_i^{+} \\ \Psi(\vec{r}) &= \sum_i \varphi_i(\vec{r})\, a_i \end{aligned}\right\}$$

Then

$$\Psi^{+}(\vec{r})|0\rangle = \sum_i |i\rangle\langle i|\vec{r}\rangle = |\vec{r}\rangle$$

i.e. $\Psi^{+}(\vec{r})$ creates and $\Psi(\vec{r})$ annihilates a particle at position $\vec{r}$.

Properties of the field operators:

(i) Commutation relations: To simplify notation we introduce

$$\left.\begin{aligned} [A,B]_- &= AB - BA = [A,B] \\ [A,B]_+ &= AB + BA = \{A,B\} \end{aligned}\right\}$$

Then it follows straightforwardly from the relations for the a_i^{+}, a_i that

$$[\Psi(\vec{r}), \Psi(\vec{r}\,')]_{\pm} = 0$$

$$[\Psi^{+}(\vec{r}), \Psi^{+}(\vec{r}\,')]_{\pm} = 0$$

$$[\Psi(\vec{r}), \Psi^{+}(\vec{r}\,')]_{\pm} = \sum_{ij} \varphi_i(\vec{r})\, \varphi_j^{*}(\vec{r}\,')\, \underbrace{[a_i, a_j^{+}]_{\pm}}_{=\delta_{ij}}$$

$$= \sum_i \varphi_i(\vec{r})\varphi_i^*(\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

(ii) Kinetic energy: $T = \sum_{ij} t_{ij}\, a_i^\dagger a_j$ with

$$t_{ij} = \int d^3r\, \varphi_i^*(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\varphi_j(\vec{r}) =$$

$$= \frac{\hbar^2}{2m}\int d^3r\, (\nabla\varphi_i^*)\cdot(\nabla\varphi_j)$$

$$\Rightarrow T = \frac{\hbar^2}{2m}\sum_{ij}\int d^3r\, (\nabla\varphi_i^* a_i^\dagger)\cdot(\nabla\varphi_j a_j)$$

$$= \frac{\hbar^2}{2m}\int d^3r \left(\nabla\sum_i \varphi_i^* a_i^\dagger\right)\cdot\left(\nabla\sum_j \varphi_j a_j\right) =$$

$$= \frac{\hbar^2}{2m}\int d^3r\, (\nabla\psi^\dagger)\cdot(\nabla\psi)$$

T

(iii) Single particle potential: $V^{(1)} = \sum_{ij} V_{ij}^{(1)} a_i^\dagger a_j$

$$V_{ij}^{(1)} = \int d^3r\, \varphi_i^*(\vec{r})\,\varphi_j(\vec{r})\,v^{(1)}(\vec{r})$$

$$\Rightarrow V^{(1)} = \int d^3r\, v^{(1)}(\vec{r})\,\psi^\dagger(\vec{r})\psi(\vec{r})$$

$V^{(1)}$

(iv) Two-particle interaction:

$$\langle ij|V^{(2)}|k,l\rangle = \int d^3r\int d^3r'\, \varphi_i^*(\vec{r})\varphi_j^*(\vec{r}\,')\times$$

$$\times \; v^{(2)}(\vec{r},\vec{r}')\,\phi_k(\vec{r})\,\phi_l(\vec{r}')$$

$$\Rightarrow \quad V^{(2)} = \frac{1}{2}\sum_{ijkl} \langle ij | V^{(2)} | kl \rangle \, a_i^+ a_j^+ a_l a_k$$

$$= \frac{1}{2}\int d^3r \int d^3r' \, v^{(2)}(\vec{r},\vec{r}')\,\psi^+(\vec{r})\,\psi^+(\vec{r}')\,\psi(\vec{r}')\,\psi(\vec{r})$$

(v) Particle density: $n(\vec{r}) := \sum_\alpha \delta(\vec{r}-\vec{r}_\alpha)$

where $\vec{r}_\alpha$ is the position operator of the α'th particle. Thus the single particle operator of particle α is $\delta(\vec{r}-\vec{r}_\alpha)$ with matrix elements

$$n_{ij}(\vec{r}) = \int d^3r_\alpha \, \phi_i^*(\vec{r}_\alpha)\,\delta(\vec{r}-\vec{r}_\alpha)\,\phi_j(\vec{r}_\alpha) = \phi_i^*(\vec{r})\,\phi_j(\vec{r})$$

$$n \quad \Rightarrow \quad n(\vec{r}) = \sum_{ij} \phi_i^*(\vec{r})\,\phi_j(\vec{r})\,a_i^+ a_j = \psi^+(\vec{r})\,\psi(\vec{r})$$

and the total particle number operator is

$$\hat{N} = \int d^3r \, \psi^+(\vec{r})\,\psi(\vec{r})$$

Remark on the term "second quantization":

The relations T, V', n follow (formally) from the corresponding expressions of single particle quantum mechanics

$$\begin{cases} \langle T \rangle = \frac{\hbar^2}{2m} \int d^3r \, |\nabla \psi|^2 & \text{expectation value of kinetic energy} \\ \langle V^{(1)} \rangle = \int d^3r \, v^{(1)}(\vec{r}) \, |\psi(\vec{r})|^2 & \text{expectation value of potential energy} \\ n(\vec{r}) = |\psi(\vec{r})|^2 & \text{probability density} \end{cases}$$

by replacing the single particle wave function by the field operators.

---

b) <u>Equations of motion</u>

We derive an equation of motion for the field operator in the <u>Heisenberg picture</u>:

$$\psi(\vec{r}, t) := \psi_H(\vec{r}, t) = e^{\frac{i}{\hbar} tH} \, \psi(\vec{r}) \, \underbrace{e^{-\frac{i}{\hbar} tH}}_{U}$$

with the Hamiltonian

$$H = \frac{\hbar^2}{2m} \int d^3r \, (\nabla \psi^\dagger) \cdot (\nabla \psi) \; +$$

$$+ \int d^3r \, v^{(1)}(\vec{r}) \, \psi^\dagger(\vec{r}) \psi(\vec{r}) \; +$$

$$+ \frac{1}{2} \int d^3r \int d^3r' \, V^{(2)}(\vec{r},\vec{r}') \, \psi^+(\vec{r}) \, \psi^+(\vec{r}') \, \psi(\vec{r}') \, \psi(\vec{r})$$

Since H commutes with N, we have

$$i\hbar \frac{d\psi}{dt} = -[H,\psi]_- = -e^{\frac{i}{\hbar}tH}[H,\psi(\vec{r})]_- e^{-\frac{i}{\hbar}tH}$$

with $\psi(\vec{r}) = \psi(\vec{r},0)$. The following relations are useful:

$$[AB,C]_- = A[B,C]_\pm \mp [A,C]_\pm B$$

(see exercise 1.1. a) for proof). With this:

$$[T,\psi(\vec{r})]_- = \frac{\hbar^2}{2m} \int d^3r' \left[\nabla'\psi^+(\vec{r}') \cdot \nabla'\psi(\vec{r}'), \psi(\vec{r})\right]_-$$

$$= \frac{\hbar^2}{2m} \int d^3r' \Big(\nabla'\psi^+ \cdot \underbrace{[\nabla'\psi(\vec{r}'), \psi(\vec{r})]_\pm}_{=0} \mp$$

$$\mp \underbrace{[\nabla'\psi^+(\vec{r}'), \psi(\vec{r})]_\pm} \cdot \nabla'\psi(\vec{r}') \Big)$$

$$= \nabla'[\psi^+(\vec{r}'), \psi(\vec{r})]_\pm = \pm \nabla'\delta(\vec{r}'-\vec{r})$$

upper: fermions lower: bosons

$$\Rightarrow [T,\psi(\vec{r})] = \frac{\hbar^2}{2m} \int d^3r' \left(-\nabla'\delta(\vec{r}'-\vec{r})\right) \cdot \nabla'\psi(\vec{r}') =$$

$$= \frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r})$$

In the same way we show that

$$[V^{(1)}, \psi(\vec{r})]_- = - v^{(1)}(\vec{r}) \psi(\vec{r})$$

For the two-particle contribution we need to compute

$$\frac{1}{2} \int d^3r' \int d^3r'' \, v^{(2)}(\vec{r}', \vec{r}'') \underbrace{[\psi^+(\vec{r}')\psi^+(\vec{r}'')\psi(\vec{r}'')\psi(\vec{r}'), \psi(\vec{r})]_-}$$

$$= \psi^+(\vec{r}')\psi^+(\vec{r}'')\psi(\vec{r}'')\psi(\vec{r}')\psi(\vec{r}) -$$

$$- \psi(\vec{r})\psi^+(\vec{r}')\psi^+(\vec{r}'')\psi(\vec{r}'')\psi(\vec{r}') =$$

$$= \underbrace{[\psi^+(\vec{r}')\psi^+(\vec{r}''), \psi(\vec{r})]_-} \psi(\vec{r}'')\psi(\vec{r}')$$

$$= \psi^+(\vec{r}')[\psi^+(\vec{r}''), \psi(\vec{r})]_\pm \mp [\psi^+(\vec{r}'), \psi(\vec{r})]_\pm \psi^+(\vec{r}'')$$

$$= \pm \psi^+(\vec{r}')\delta(\vec{r}'' - \vec{r}) - \delta(\vec{r}' - \vec{r})\psi^+(\vec{r}'')$$

$$\Rightarrow \frac{1}{2} \int d^3r' \int d^3r'' \, v^{(2)}(\vec{r}', \vec{r}'') [\psi^+(\vec{r}')\psi^+(\vec{r}'')\psi(\vec{r}'')\psi(\vec{r}'), \psi(\vec{r})]_-$$

$$= \pm \frac{1}{2} \int d^3r' \, v^{(2)}(\vec{r}', \vec{r}) \psi^+(\vec{r}') \underbrace{\psi(\vec{r})\psi(\vec{r}')}_{= \mp \psi(\vec{r}')\psi(\vec{r})} +$$

$$-\frac{1}{2}\int d^3r''\, v^{(2)}(\vec{r},\vec{r}'')\, \psi^+(\vec{r}'')\,\psi(\vec{r}'')\,\psi(\vec{r})$$

$$= -\int d^3\vec{r}'\, v^{(2)}(\vec{r},\vec{r}')\, \psi^+(\vec{r}')\,\psi(\vec{r}')\,\psi(\vec{r})$$

where it has been assumed that the interaction is symmetric, $v^{(2)}(\vec{r},\vec{r}') = v^{(2)}(\vec{r}',\vec{r})$.

To summarize:

$$i\hbar\frac{\partial\psi}{\partial t} = \underbrace{\left[-\frac{\hbar^2}{2m}\nabla^2 + v^{(1)}(\vec{r})\right]}_{\text{single particle Hamiltonian } H^{(1)}}\psi(\vec{r},t)$$

$$+\int d^3r'\, v^{(2)}(\vec{r},\vec{r}')\, e^{\frac{i}{\hbar}tH}\,\psi^+(\vec{r}')\,\psi(\vec{r}')\,\psi(\vec{r})\,e^{-\frac{i}{\hbar}tH}$$

Using that

$$\left.\begin{aligned}\psi(\vec{r}) &= e^{-\frac{i}{\hbar}tH}\,\psi(\vec{r},t)\,e^{\frac{i}{\hbar}tH}\\ \psi^+(\vec{r}) &= e^{-\frac{i}{\hbar}tH}\,\psi^+(\vec{r},t)\,e^{\frac{i}{\hbar}tH}\end{aligned}\right\}$$

the interaction term becomes

$$\int d^3r'\, v^{(2)}(\vec{r},\vec{r}')\,\psi^+(\vec{r}',t)\,\psi(\vec{r}',t)\,\psi(\vec{r},t)$$

c) Momentum representation

In the momentum representation the eigenfunctions are plane waves. To allow for normalization, we assume confinement of the particle to a volume

$$\left.\begin{aligned} V &= L_x L_y L_z \\ \Rightarrow \quad \varphi_{\vec{k}}(\vec{r}) &= \frac{1}{\sqrt{V}} \exp[i \vec{k} \cdot \vec{r}] \end{aligned}\right\}$$

with periodic boundary conditions

$$\varphi_{\vec{k}}(\vec{r} + L_x \vec{e}_x) = \varphi_{\vec{k}}(\vec{r}) \quad \text{etc.}$$

$\Rightarrow$ permissible momentum vectors are of the form

$$\vec{k} = 2\pi \left(\frac{\nu_x}{L_x}, \frac{\nu_y}{L_y}, \frac{\nu_z}{L_z} \right), \quad \nu_{x,y,z} \in \mathbb{Z}$$

These states are orthonormal,

$$\int d^3r \, \varphi^*_{\vec{k}}(\vec{r}) \varphi_{\vec{k}'}(\vec{r}) = \delta_{\vec{k},\vec{k}'}$$

Matrix elements

(i) Kinetic energy: $t_{\vec{k},\vec{k}'} = -\frac{\hbar^2}{2m} \int d^3r \, \varphi^*_{\vec{k}} \nabla^2 \varphi_{\vec{k}'}$

$$= \frac{\hbar^2}{2m} |\vec{k}|^2 \int d^3r \, \varphi^*_{\vec{k}}(\vec{r}) \varphi_{\vec{k}'}(\vec{r}) = \frac{\hbar^2}{2m} |\vec{k}|^2 \delta_{\vec{k},\vec{k}'}$$

(ii) Single-particle potential:

$$\underline{V^{(1)}_{\vec{k},\vec{k}'}} = \frac{1}{V} \int d^3\vec{r}\; v^{(1)}(\vec{r})\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} =$$

$$= \underline{\frac{1}{V}\, \hat{v}^{(1)}(\vec{k}-\vec{k}')}$$

where $\hat{v}^{(1)}(\vec{q}) = \int d^3r\; e^{-i\vec{q}\cdot\vec{r}}\, v^{(1)}(\vec{r})$

is the <u>Fourier transform</u> of the potential.

(iii) Two-particle interaction:

We assume that the interaction potential is translationally invariant, i.e.

$$v^{(2)}(\vec{r},\vec{r}') = v^{(2)}(\vec{r}-\vec{r}')$$

Then the matrix element of interest is

$$\langle \vec{k}', \vec{q}' \,|\, V^{(2)} \,|\, \vec{k}, \vec{q} \rangle =$$

$$= \frac{1}{V^2} \int d^3r \int d^3r'\; e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\; e^{i(\vec{q}-\vec{q}')\cdot\vec{r}'} \times$$

$$\times \underbrace{v^{(2)}(\vec{r}-\vec{r}')}$$

$$= \frac{1}{V} \sum_{\vec{p}} \hat{v}^{(2)}(\vec{p})\, e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}$$

<u>inverse Fourier transform</u>

$$\Rightarrow \langle \vec{k}', \vec{q}' | V^{(2)} | \vec{k}, \vec{q} \rangle =$$

$$= \frac{1}{V^3} \sum_{\vec{p}} \hat{v}^{(2)}(\vec{p}) \underbrace{\int d^3 r \, e^{i(\vec{k}-\vec{k}'+\vec{p})\cdot\vec{r}}}_{V \delta_{\vec{k}-\vec{k}', -\vec{p}}} \times$$

$$\times \underbrace{\int d^3 r' \, e^{i(\vec{q}-\vec{q}'-\vec{p})\cdot\vec{r}'}}_{= V \delta_{\vec{q}-\vec{q}', \vec{p}}} =$$

$$= \underline{\frac{1}{V} \sum_{\vec{p}} \hat{v}^{(2)}(\vec{p}) \, \delta_{\vec{k}-\vec{k}', -\vec{p}} \, \delta_{\vec{q}-\vec{q}', \vec{p}}}$$

It follows that the Hamiltonian in second quantized form reads

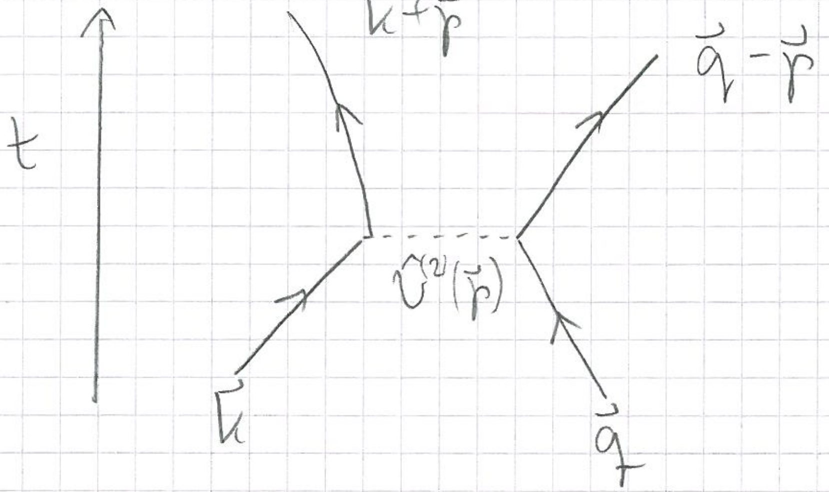
$$H = \sum_{\vec{k}, \vec{k}'} \left[\frac{\hbar^2}{2m} |\vec{k}|^2 \delta_{\vec{k}, \vec{k}'} + \frac{1}{V} \hat{v}^{(1)}(\vec{k}-\vec{k}') \right] a^+_{\vec{k}'} a_{\vec{k}} +$$

$$+ \frac{1}{2V} \sum_{\vec{k}, \vec{q}, \vec{p}} \hat{v}^{(2)}(\vec{p}) \, a^+_{\vec{k}+\vec{p}} \, a^+_{\vec{q}-\vec{p}} \, a_{\vec{q}} \, a_{\vec{k}}$$

Interpretation of the interaction term:

Two particles with momenta $(\vec{k}, \vec{q})$ are replaced by two particles with momenta $(\vec{k}+\vec{p}, \vec{q}-\vec{p})$. This conserves the total momentum, and the transferred momentum $\vec{p}$ is provided by the interaction.

Such interactions are conveniently represented by Feynman diagrams:



(iv) Density operator[1]: $n(\vec{r}) = \psi^\dagger(\vec{r})\,\psi(\vec{r})$

$$\left.\begin{aligned} \psi(\vec{r}) &= \frac{1}{\sqrt{V}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\, a_{\vec{k}}\,, \\ \psi^\dagger(\vec{r}) &= \frac{1}{\sqrt{V}} \sum_{k} e^{-i\vec{k}\cdot\vec{r}}\, a^\dagger_{\vec{k}} \end{aligned}\right\}$$

$$\Rightarrow \quad n(\vec{r}) = \frac{1}{V} \sum_{\vec{k},\vec{k}'} e^{i\vec{r}\cdot(\vec{k}-\vec{k}')}\, a^\dagger_{\vec{k}'}\, a_{\vec{k}}$$

---

[1] Note that this is <u>not</u> the same as ρ