where E_0 is the ground state energy, $\varepsilon_{\vec{k}} \geq 0$ are the quasiparticle energies and the $b_{\vec{k}}, b_{\vec{k}}^+$ are bosonic operators.

To find the b_k, b_k^+ we make a linear ansatz

$$\left| \begin{aligned} a_{\vec{k}} &= u_{\vec{k}} b_{\vec{k}} + v_{\vec{k}} b_{-\vec{k}}^+ \\ a_{\vec{k}}^+ &= u_{\vec{k}} b_{\vec{k}}^+ + v_{\vec{k}} b_{-\vec{k}} \end{aligned} \right\}$$

with real coefficients $u_{\vec{k}}, v_{\vec{k}} \in \mathbb{R}$,

$u_{\vec{k}} = u_{-\vec{k}}$,

$v_{\vec{k}} = v_{-\vec{k}}$

We know from exercise 4.2 that the $b_{\vec{k}}, b_{\vec{k}}^+$ are bosonic operators provided

$$u_{\vec{k}}^2 - v_{\vec{k}}^2 = 1 \qquad (1)$$

Inserting into H yields

$$H = \frac{1}{2} \hat{v}^{(2)}(0) \frac{N^2}{V} + \sum_{\vec{k} \neq 0} \left(\frac{\hbar^2 |\vec{k}|^2}{2m} + n \hat{v}^{(2)}(\vec{k}) \right) \Big[u_{\vec{k}}^2 b_{\vec{k}}^+ b_{\vec{k}}$$

$$+ v_{\vec{k}}^2 b_{\vec{k}} b_{\vec{k}}^+ + \underline{u_{\vec{k}} v_{\vec{k}} (b_{\vec{k}}^+ b_{-\vec{k}}^+ + b_{\vec{k}} b_{-\vec{k}})} \Big] +$$

$$+ \frac{n}{2} \sum_{\vec{k} \neq 0} \hat{v}^{(2)}(\vec{k}) \Big[\underline{(u_{\vec{k}}^2 + v_{\vec{k}}^2)(b_{\vec{k}}^+ b_{-\vec{k}}^+ + b_{\vec{k}} b_{-\vec{k}})} +$$

$$+ 2 u_{\vec{k}} v_{\vec{k}} (b_{\vec{k}}^+ b_{\vec{k}} + b_{\vec{k}} b_{\vec{k}}^+) \Big]$$

To achieve diagonalization, the underlined terms must vanish:

(2) $$\left(\frac{\hbar^2|\vec{k}|^2}{2m} + n\hat{U}^{(2)}(k)\right) u_{\vec{k}} v_{\vec{k}} + \frac{n}{2}\hat{U}^{(2)}(\vec{k})\left(u_{\vec{k}}^2 + v_{\vec{k}}^2\right) = 0$$

$\Rightarrow$ two coupled quadratic equations for each $\vec{k}$.

The solution reads (cf. exercise 5.2)

$$\left.\begin{aligned} u_{\vec{k}}^2 &= \frac{1}{2}\left(\frac{w_{\vec{k}}}{\varepsilon_{\vec{k}}} + 1\right), \quad v_{\vec{k}}^2 = \frac{1}{2}\left(\frac{w_{\vec{k}}}{\varepsilon_{\vec{k}}} - 1\right) \\ u_{\vec{k}} v_{\vec{k}} &= -\frac{n}{2}\frac{\hat{U}^{(2)}(\vec{k})}{\varepsilon_{\vec{k}}} \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{with} \quad w_{\vec{k}} &:= \frac{\hbar|\vec{k}|^2}{2m} + n\hat{U}^{(2)}(\vec{k}) \\ \varepsilon_{\vec{k}} &= \sqrt{w_{\vec{k}}^2 - n^2\hat{U}^{(2)}(\vec{k})^2} = \sqrt{\left(\frac{\hbar^2|\vec{k}|^2}{2m}\right)^2 + \frac{\hbar^2|\vec{k}|^2 n\hat{U}^{(2)}(\vec{k})}{m}} \end{aligned}\right\}$$

It follows that

$$H = \frac{1}{2}\hat{U}^{(2)}(0)\frac{N^2}{V} + \sum_{\vec{k}\neq 0} w_{\vec{k}}\left[u_{\vec{k}}^2 b_{\vec{k}}^\dagger b_{\vec{k}} + v_{\vec{k}}^2 \underbrace{b_{\vec{k}} b_{\vec{k}}^\dagger}_{=1+b_{\vec{k}}^\dagger b_{\vec{k}}}\right]$$

$$+ n\sum_{\vec{k}\neq 0}\hat{U}^{(2)}(\vec{k}) u_{\vec{k}} v_{\vec{k}}\left[b_{\vec{k}}^\dagger b_{\vec{k}} + \underbrace{b_{\vec{k}} b_{\vec{k}}^\dagger}_{=1+b_{\vec{k}}^\dagger b_{\vec{k}}}\right] =$$

$$= \frac{1}{2}\hat{U}^{(2)}(0)\frac{N^2}{V} + \sum_{\vec{k}\neq 0}\left(w_{\vec{k}} v_{\vec{k}}^2 + n\hat{U}^{(2)}(\vec{k}) u_{\vec{k}} v_{\vec{k}}\right) +$$

$$+ \sum_{\vec{k}\neq 0}\left[w_{\vec{k}}\left(u_{\vec{k}}^2 + v_{\vec{k}}^2\right) + 2n\hat{U}^{(2)}(\vec{k}) u_{\vec{k}} v_{\vec{k}}\right] b_{\vec{k}}^\dagger b_{\vec{k}} =$$

$$= \underbrace{\frac{1}{2}\,\hat{v}^{(2)}(0)\,\frac{N^2}{V} + \frac{1}{2}\sum_{\vec{k}\neq 0}\left(\varepsilon_{\vec{k}} - w_{\vec{k}}\right)}_{=:\,E_0 \text{ ground state energy}} +$$

$$+ \underbrace{\sum_{\vec{k}\neq 0} \varepsilon_{\vec{k}}\, b^{+}_{\vec{k}}\, b_{\vec{k}}}_{\text{quasiparticle excitations}}$$

---

c) <u>Ground state properties</u>

The ground state $|\Phi_0\rangle$ has energy E_0 and is defined by

$$b_{\vec{k}}\,|\Phi_0\rangle = 0 \quad \forall\ \vec{k}$$

To check the validity of the quadratic approximation we first compute the number of particles N' that are not in the condensate $\vec{k} = 0$.

$$\underline{N'} = \sum_{\vec{k}\neq 0} \langle \Phi_0 |\, a^{+}_{\vec{k}}\, a_{\vec{k}}\, | \Phi_0 \rangle =$$

$$= \sum_{\vec{k}\neq 0} \langle \Phi_0 |\,(u_{\vec{k}}\, b^{+}_{\vec{k}} + v_{\vec{k}}\, b_{-\vec{k}})(u_{k}\, b_{\vec{k}} + v_{\vec{k}}\, b^{+}_{-\vec{k}})\,|\Phi_0\rangle$$

$$= \sum_{\vec{k}\neq 0} v_{\vec{k}}^{2}\, \langle \Phi_0 |\, \underbrace{b_{\vec{k}}\, b^{+}_{\vec{k}}}_{1 + b^{+}_{\vec{k}} b_{\vec{k}}}\, | \Phi_0 \rangle = \underline{\sum_{\vec{k}\neq 0} v_{\vec{k}}^{2}}$$

Consider specifically a repulsive contact interaction of strength λ:

$$v^{(2)}(\vec{r}) = \lambda\, \delta^{(3)}(\vec{r}) \quad \Rightarrow \quad \hat{v}^{(2)}(\vec{k}) = \lambda > 0$$

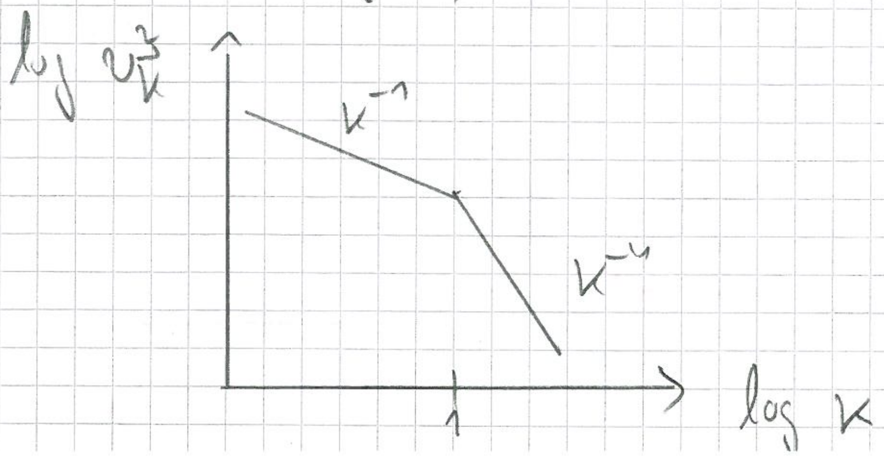
$$\Rightarrow \quad v_{\vec{k}}^2 = \frac{1}{2}\left(\frac{1+\kappa^2}{\sqrt{2\kappa^2+\kappa^4}} - 1\right) \quad , \quad \kappa = |\vec{k}|\, l$$

where κ is dimensionless and the length scale l is defined by

$$l^2 = \frac{\hbar^2}{2mn\lambda} \quad \Longleftrightarrow \quad \underbrace{\frac{\hbar^2}{2m\, l^2}}_{\substack{\text{kinetic energy}\\ \text{of particle confined}\\ \text{to scale } l}} = \underbrace{n\lambda}_{\substack{\text{potential energy}\\ \text{per particle}}}$$

$v_{\vec{k}}^2 = \langle \Phi_0 | a_{\vec{k}}^\dagger a_{\vec{k}} | \Phi_0 \rangle$ is the momentum distribution in the ground state, and it behaves as

$$v_{\vec{k}}^2 \simeq \begin{cases} 2^{-3/2}\, (kl)^{-1} \quad , & \kappa = kl \ll 1 \\[2ex] \frac{1}{4}\, (kl)^{-4} \quad , & \kappa = kl \gg 1 \end{cases}$$



Replacing the sum over $\vec{k}$ by an integral we find

$$\underline{N'} = \sum_{\vec{k} \neq 0} v_{\vec{k}}^2 \simeq \frac{V}{(2\pi)^3} \int d^3k \; v_k^3 =$$

$$= \frac{1}{(2\pi)^3} \frac{V}{\ell^3} \cdot \int d^3k \cdot \frac{1}{2} \left(\frac{1+k^2}{\sqrt{2k^2+k^4}} - 1 \right) =$$

$$= \frac{1}{4\pi^2} \frac{V}{\ell^3} \int_0^\infty dk \, k^2 \left(\frac{1+k^2}{\sqrt{2k^2+k^4}} - 1 \right) = \underline{\frac{1}{3 \cdot 2^{7/2} \pi^2} \cdot \frac{V}{\ell^3}}$$

$\Rightarrow$ density of particles outside the condensate

$$\underline{n' = \frac{N'}{V} = \frac{1}{3\pi^2} \cdot \left(\frac{m n \lambda}{\hbar^2} \right)^{3/2}}$$

The condition for the validity of the weakly interacting approximation is

$$\underline{\frac{n'}{n} = \frac{1}{3\pi^2} \left(\frac{m}{\hbar} \right)^{3/2} \left(n \lambda^3 \right)^{1/2} \ll 1}$$

which implies low density and/or weak interaction.

Digression on Bose-Einstein condensation in the ideal Bose gas

In the ideal Bose gas, the ground state is occupied by a macroscopic number of particles N_0 for temperatures below the Bose temperature T_B.

The number of particles in excited states is given by

$$N' = \sum_{\vec{k}\neq 0} n_{\vec{k}} \quad , \quad n_{\vec{k}} = \underbrace{\frac{1}{e^{\beta E_{\vec{k}}}-1}}_{\substack{\text{Bose-Einstein distribution} \\ \text{with chemical potential } \mu=0}} \quad , \quad \beta = \frac{1}{k_B T}$$

where $E_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}$ for free, massive particles.

Replacing the sum by an integral thus yields

$$N' = \frac{V}{2\pi^2}\left(\frac{2m\,k_B T}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty dx\, \frac{x^2}{e^{x^2}-1}}_{\cong 1.157}$$

$$\cong 0.166 \cdot V \cdot \left(\frac{m\,k_B T}{\hbar^2}\right)^{3/2}$$

Up to the value of the numerical prefactor, this is of the same form as the result for the weakly interacting Bose gas provided the energy scale $n\lambda$ is replaced by $k_B T$.

In fact the general form of this expression can be deduced by dimensional analysis:

(i) We expect that N' is proportional to the volume V, and depends on the energy scale ε:

$$N' = V \cdot f(\varepsilon)$$

where $f(\varepsilon)$ has the dimension

$$(\text{volume})^{-1} = (\text{length})^{-3}$$

(ii) In addition to ε, $f(\varepsilon)$ can depend only on the mass m of the particles and on $\hbar$.

(iii) From the form of the kinetic energy operator

$$t = -\frac{\hbar^2}{2m}\nabla^2$$

we obtain a length scale l by setting

$$\frac{\hbar^2}{m l^2} = \varepsilon \quad \Rightarrow \quad l = \left(\frac{\hbar^2}{m\varepsilon}\right)^{1/2}$$

$$\Rightarrow \quad f(\varepsilon) \sim l^{-3} = \left(\frac{m\varepsilon}{\hbar^2}\right)^{3/2}$$

Ground state energy:

$$E_0 = \frac{1}{2}\hat{U}^{(2)}(0)\frac{N^2}{V} + \frac{1}{2}\sum_{\vec{k}\neq 0}(\varepsilon_{\vec{k}} - w_{\vec{k}})$$

with $\varepsilon_{\vec{k}} = \sqrt{w_{\vec{k}}^2 - n^2\hat{U}^{(2)}(\vec{k})^2} < w_{\vec{k}}$

$\Rightarrow$ expulsion of particles from the condensate lowers the energy.

For the contact potential

$$w_{\vec{k}} - \varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} + n^2\lambda^2 - \sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + 2\frac{\hbar^2 k^2}{2m}n\lambda}$$

$$= n\lambda\left[1 + \kappa^2 - \sqrt{\kappa^4 + 2\kappa^2}\right], \quad \kappa = kl$$

$$\simeq \begin{cases} n\lambda & kl \ll 1 \\ \frac{1}{2}n\lambda(kl)^{-2} & kl \gg 1 \end{cases}$$

The latter result implies that the integral

$$\sum_{\vec{k}\neq 0}(w_{\vec{k}} - \varepsilon_{\vec{k}}) \simeq \frac{V}{(2\pi)^3}\cdot 4\pi\int_0^\infty dk\, k^2(w_{\vec{k}} - \varepsilon_{\vec{k}})$$

diverges at large $|\vec{k}|$, and hence $E_0 = -\infty$ for this potential. A finite ground state energy is obtained for interaction potentials of finite range such as

$$V^{(2)}(\vec{r}) = \lambda' \Theta(R - |\vec{r}|) = \begin{cases} \lambda' & |\vec{r}| \leq R \\ 0 & |\vec{r}| > R \end{cases}$$

---

d) Quasiparticles

The excitations from the ground state $|\Phi_0\rangle$ are called quasiparticles, and are generated by applying $b^+_{\vec{k}}$ to the ground state:

$$|\vec{k}\rangle_{qp} := b^+_{\vec{k}} |\Phi_0\rangle$$

It follows from the form of the Hamiltonian H that a quasiparticle with wave vector $\vec{k}$ has energy $\varepsilon_{\vec{k}}$. We now show that quasiparticles are also momentum eigenstates.

The momentum operator is

$$\vec{P} = \sum_{\vec{k}} \hbar \vec{k}\, a^+_{\vec{k}} a_{\vec{k}} = \sum_{\vec{k}} \hbar \vec{k} (u_{\vec{k}} b^+_{\vec{k}} + v_{\vec{k}} b_{-\vec{k}})$$

$$\times (u_{\vec{k}} b_{\vec{k}} + v_{\vec{k}} b^+_{-\vec{k}}) =$$

$$= \sum_{\vec{k}} \hbar \vec{k}\, u^2_{\vec{k}} b^+_{\vec{k}} b_{\vec{k}} + \sum_{\vec{k}} \hbar \vec{k}\, u_{\vec{k}} \cancel{v_{\vec{k}}} b^+_{\vec{k}} b^+_{-\vec{k}} +$$

$$+ \sum_{\vec{k}} \hbar\vec{k}\, \cancel{u_{\vec{k}} v_{\vec{k}}\, b_{-\vec{k}}^{+} b_{\vec{k}}^{+}} + \sum_{\vec{k}} \hbar\vec{k}\, v_{\vec{k}}^2\, b_{-\vec{k}}\, b_{-\vec{k}}^{+}$$

Because $u_{\vec{k}} = u_{-\vec{k}}$, $v_{\vec{k}} = v_{-\vec{k}}$, the second and third sum vanish.

$$\Rightarrow \vec{P} = \sum_{\vec{k}} \hbar\vec{k}\, u_{\vec{k}}^2\, b_{\vec{k}}^{+} b_{\vec{k}} - \sum_{\vec{k}} \hbar\vec{k}\, v_{\vec{k}}^2\, \underbrace{b_{\vec{k}}\, b_{\vec{k}}^{+}}_{= 1 + b_{\vec{k}}^{+} b_{\vec{k}}}$$

$$= \sum_{\vec{k}} \hbar\vec{k}\, \underbrace{(u_{\vec{k}}^2 - v_{\vec{k}}^2)}_{=1}\, b_{\vec{k}}^{+} b_{\vec{k}} - \underbrace{\sum_{\vec{k}} \hbar\vec{k}\, v_{\vec{k}}^2}_{=0}$$

$$= \sum_{\vec{k}} \hbar\vec{k}\, b_{\vec{k}}^{+} b_{\vec{k}}$$

It follows that

$$\vec{P}\,|\vec{k}\rangle_{qp} = \sum_{\vec{k}'} \hbar\vec{k}'\, \underbrace{b_{\vec{k}'}^{+} b_{\vec{k}'}\, b_{\vec{k}}^{+}\, |\Phi_0\rangle}$$

$$= b_{\vec{k}'}^{+}\left(b_{\vec{k}}^{+} b_{\vec{k}'} + \delta_{\vec{k},\vec{k}'} \right)$$

$$\Rightarrow \vec{P}\,|\vec{k}\rangle_{qp} = \hbar\vec{k}\,|\vec{k}_{qp}\rangle$$

The quasiparticle has a sharp momentum $\hbar\vec{k}$ and energy $\varepsilon_{\vec{k}}$.

The relation between energy (frequency) and momentum (wave number) of a particle or quasiparticle is called a dispersion relation.

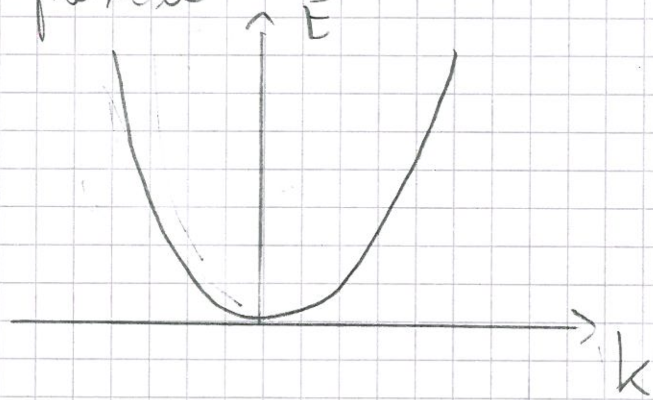
Dispersion of a wave means that the group velocity differs from the phase velocity:

$$v_p = \frac{\omega(k)}{k} \neq v_g = \frac{d\omega}{dk} \Rightarrow \omega(k) \text{ is non linear}$$

Examples of known dispersion relations:

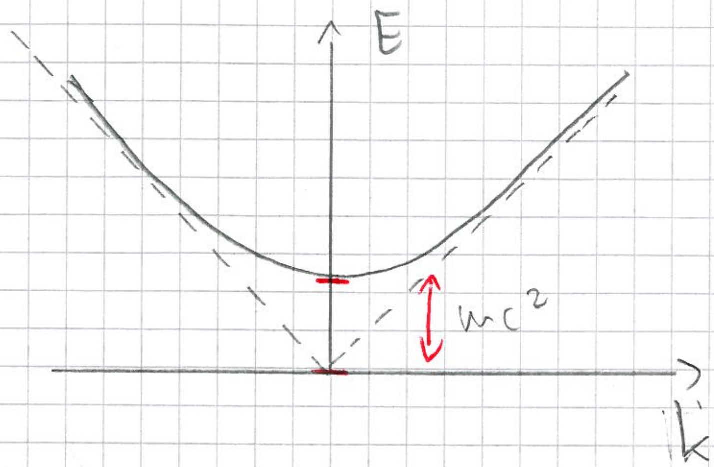
(i) Non relativistic massive particle

$$E(\vec{k}) = \frac{\hbar^2 |\vec{k}|^2}{2m}$$



(ii) Relativistic massive particle

$$E(\vec{k}) = \sqrt{m^2 c^4 + \hbar^2 c^2 |\vec{k}|^2}$$



$$E(\vec{k}) \simeq \begin{cases} mc^2 + \dfrac{\hbar^2 k^2}{2m}, & k \to 0 \\ \hbar c |k|, & k \to \infty \end{cases}$$

(iii) Relativistic massless particle:

$$E(\vec{k}) = \hbar c |\vec{k}| \implies \text{no dispersion}$$

(iv) Quasiparticles in the weakly interacting Bose gas:

$$\varepsilon_{\vec{k}} = \sqrt{\underbrace{\left(\frac{\hbar^2 |\vec{k}|^2}{2m}\right)^2}_{\sim |\vec{k}|^4} + \underbrace{\frac{\hbar^2 |\vec{k}|^2 \, n \, \hat{U}^{(2)}(\vec{k})}{m}}_{\sim |\vec{k}|^2}}$$

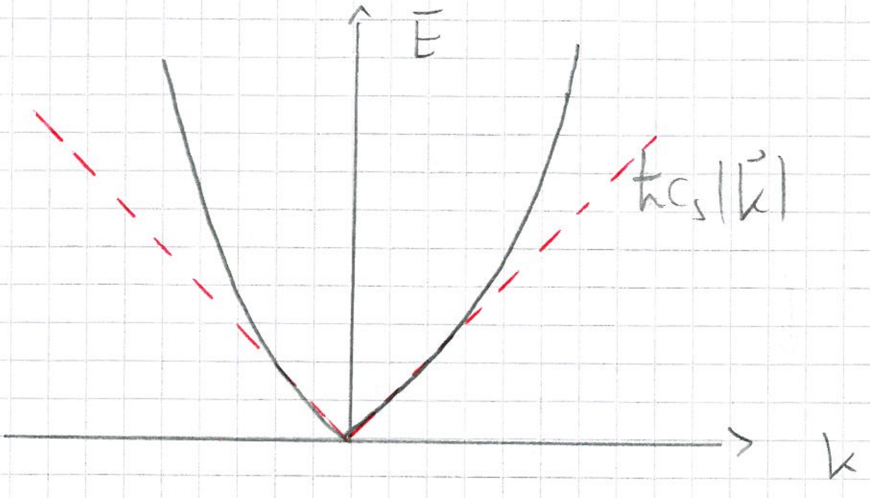
$|\vec{k}| \to 0$:
$$\varepsilon_{\vec{k}} \simeq \sqrt{\frac{n \, \hat{U}^{(2)}(0)}{m}} \cdot \hbar |\vec{k}| = c_s \hbar |\vec{k}|$$

$\Rightarrow$ non-dispersive "sound wave" with speed

$$c_s = \sqrt{\frac{n \, \hat{U}^{(2)}(0)}{m}}$$

$|\vec{k}| \to \infty$:
$$\varepsilon_{\vec{k}} \simeq \frac{\hbar^2 |\vec{k}|^2}{2m} + n \, \hat{U}^{(2)}(\vec{k})$$

For the repulsive contact potential ($\hat{U}^{(2)}(\vec{k}) = \lambda > 0$)

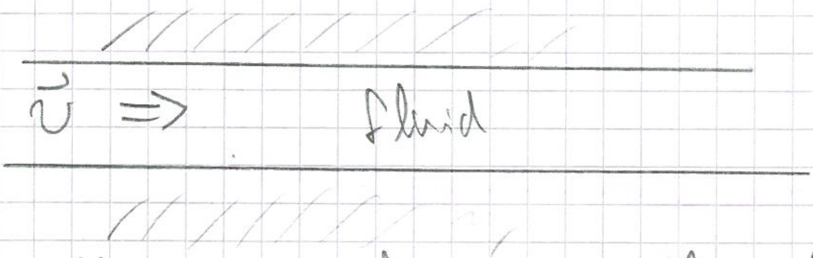


The fact that $c_s > 0$ gives rise to superfluidity (→ Section 3°)

3° Superfluidity

- ^4He liquifies at 4.2 K and enters a superfluid phase (He II) at 2.18 K.
- The superfluid flows without viscosity.
- Here we explain this behavior in terms of the weakly interacting theory. This implies in particular that the ideal Bose gas is not a superfluid.

Experimental setup: Flow in a tube



laboratory system

In the rest frame of the fluid, the tube is moving at velocity $-\vec{v}$. The two frames of reference are connected by a Galilei transformation:

Particle coordinates and momenta are

$$\left.\begin{array}{ccc} \vec{r}_i^{(0)} & & \vec{r}_i = \vec{r}_i^{(0)} + \vec{v}\,t \\ & \longrightarrow & \\ \vec{p}_i^{(0)} & & \vec{p}_i = \vec{p}_i^{(0)} + m\vec{v} \end{array}\right\}$$

rest frame $\qquad\qquad$ laboratory frame

The total momentum is

$$\vec{P} = \sum_i \vec{p}_i = \vec{P}^{(0)}_i + M\vec{v}$$

and the total energy is

$$E = \sum_i \frac{1}{2m} |\vec{p}_i|^2 + \sum_{ij} V^{(2)}(\vec{r}_i - \vec{r}_j) =$$

$$= \sum_i \frac{1}{2m} |\vec{p}^{(0)}_i + m\vec{v}|^2 + \sum_{ij} V^{(2)}(\vec{r}^{(0)}_i - \vec{r}^{(0)}_j) =$$

$$= \underbrace{\sum_i \frac{1}{2m} |\vec{p}^{(0)}_i|^2 + \sum_{ij} V^{(2)}(\vec{r}^{(0)}_i - \vec{r}^{(0)}_j)}_{= E^{(0)}} +$$

$$+ \underline{\vec{P}^{(0)} \cdot \vec{v} + \frac{1}{2} M |\vec{v}|^2}$$

Initially the fluid is in the ground state, $E^{(0)} = E_0$ and $\vec{P}^{(0)} = 0$.
<u>Friction</u> implies that the fluid generates excitations in order to lower the energy in the laboratory frame.

Generating an excitation of momentum $\vec{p}$

changes the energy and momentum as

$$\left.\begin{array}{l} E^{(0)} = E_0 \longrightarrow E_0 + \varepsilon(\vec{p}) \\ \vec{p}^{(0)} = 0 \longrightarrow \vec{p} \end{array}\right\}$$

and correspondingly in the laboratory frame

$$\left.\begin{array}{l} E = E_0 + \frac{1}{2} M (\vec{v})^2 \\ \qquad \downarrow \\ \qquad E_0 + \frac{1}{2} M (\vec{v})^2 + \varepsilon(\vec{p}) + \underbrace{\vec{p} \cdot \vec{v}}_{\substack{< 0 \text{ if } \vec{p} \\ \text{is directed along } -\vec{v}}} \end{array}\right\}$$

The energy change in the laboratory system is thus

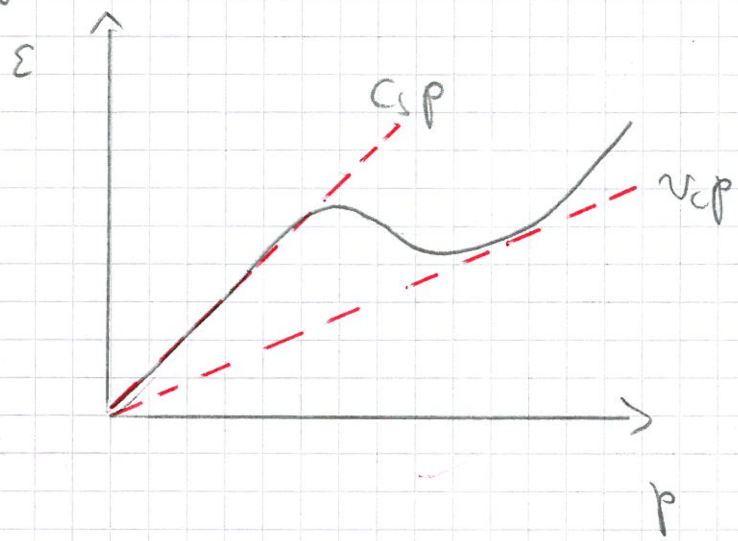
$$\Delta E = \varepsilon(\vec{p}) + \vec{p} \cdot \vec{v} \overset{!}{<} 0$$

ΔE is minimal when $\vec{p}$ and $\vec{v}$ are anti-parallel. In this case the condition reads

$$v > \min_p \frac{\varepsilon(p)}{p} =: v_c$$

Provided $v_c > 0$, dissipationless flow is possible for $v < v_c$.

For the weakly interacting Bose gas with contact potential $v_c = c_s$, but in general $v_c \leq c_s$:



Possible scenario for $v_c < c_s$

This scenario is realized by the roton minimum in He II (but note that He II is not weakly interacting).