IV. Quantum theory of light

Reminder of elementary quantum mechanics:

- photoelectric effect, Compton effect
  ↳ energy and momentum of the electromagnetic field are quantized in units $\hbar\omega$, $\hbar\vec{k}$
- Planck's radiation law
  ↳ photons are massless relativistic bosons with two independent polarizations

- Here we develop a full quantum theory of light in which the classical fields are replaced by field operators.
- This is achieved by representing the field modes as harmonic oscillators.

1° Quantization of the electromagnetic field

We formulate the Maxwell equations in vacuum using the vector potential $\vec{A}(\vec{r},t)$ in Coulomb gauge $\nabla\cdot\vec{A}=0$:

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \Box\vec{A} = 0$$

$$\Rightarrow \quad \vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla\times\vec{A}$$

The field is confined to a cavity of volume $V = L_x L_y L_z$ with periodic boundary conditions (see II. 4° c). Then $\vec{A}(\vec{r},t)$ can be represented in Fourier modes as

$$\vec{A}(\vec{r},t) = \sum_{\vec{k}} \left(\vec{A}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{r}} + \vec{A}^{*}_{\vec{k}}(t)\, e^{-i\vec{k}\cdot\vec{r}} \right)$$

with $\vec{k} = \frac{2\pi}{L}(\nu_x, \nu_y, \nu_z)$, $\nu_{x,y,z} \in \mathbb{Z}$.

The Coulomb gauge $\vec{k}\cdot\vec{A}_k = 0$ implies that the $\vec{A}_{\vec{k}}$ have two orthogonal polarization components.

The mode amplitudes satisfy

$$\frac{d^2 \vec{A}_{\vec{k}}}{dt^2} = -\omega_{\vec{k}}^2 \vec{A}_{\vec{k}} \,, \qquad \omega_{\vec{k}} = c|\vec{k}|$$

$\Rightarrow$ each mode behaves as a harmonic oscillator of frequency $\omega_{\vec{k}}$.

To make the analogy more precise consider the field energy

$$H^{1)} = \frac{1}{8\pi} \int d^3r \left(|\vec{E}|^2 + |\vec{B}|^2 \right) =$$

$$= \frac{V}{4\pi} \sum_{\vec{k}} \left\{ \frac{1}{c^2} |\dot{\vec{A}}_{\vec{k}}|^2 + |\vec{k}|^2 |\vec{A}_{\vec{k}}|^2 \right\}$$

1) Gauss units

Next introduce real scalar position and momentum coordinates $Q_{\vec{k}}$, $P_{\vec{k}}$ through the transformation

$$\vec{A}_k(t) = \sqrt{\frac{\hbar}{Vk^2}}\, e^{-i\omega_{\vec{k}} t}\left[\omega_{\vec{k}} Q_{\vec{k}} + i P_{\vec{k}}\right] \vec{e}_{\vec{k}}$$

where $\vec{e}_{\vec{k}}$ is the polarization vector of the mode, $|\vec{e}_k| = 1$. In terms of these variables

$$H = \sum_{\vec{k}} \underbrace{\frac{1}{2}\left(P_{\vec{k}}^2 + \omega_{\vec{k}}^2 Q_{\vec{k}}^2\right)}_{\text{one-dimensional harmonic oscillator of mass } m=1}$$

At this point $P_{\vec{k}}$, $Q_{\vec{k}}$ are replaced by operators with the canonical commutation relations

$$\left.\begin{aligned} &[Q_{\vec{k}}, Q_{\vec{k}'}] = [P_{\vec{k}}, P_{\vec{k}'}] = 0 \\ &[Q_{\vec{k}}, P_{\vec{k}'}] = i\hbar\, \delta_{\vec{k}\vec{k}'} \end{aligned}\right\}$$

It is well known that the harmonic oscillator problem can be solved using the ladder operators

$$\left.\begin{aligned} a_{\vec{k}} &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega_{\vec{k}} Q_{\vec{k}} + i P_{\vec{k}}\right) \\ a_{\vec{k}}^{+} &= \frac{1}{\sqrt{2\hbar\omega}}\left(\omega_{\vec{k}} Q_{\vec{k}} - i P_{\vec{k}}\right) \end{aligned}\right\}$$

with commutation relations

$$\left.\begin{aligned}[a_{\vec{k}}, a_{\vec{k}'}] &= [a^+_{\vec{k}}, a^+_{\vec{k}'}] = 0 \\ [a_{\vec{k}}, a^+_{\vec{k}'}] &= \delta_{\vec{k}\vec{k}'}\end{aligned}\right\}$$

The Hamiltonian thus becomes diagonal:

$$H = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(a^+_{\vec{k}} a_{\vec{k}} + \frac{1}{2} \right) = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \left(n_{\vec{k}} + \frac{1}{2} \right)$$

with $n_{\vec{k}} = a^+_{\vec{k}} a_{\vec{k}}$.

Interpretation:

- $a_{\vec{k}}, a^+_{\vec{k}}$ are bosonic annihilation and creation operators for photons
- Photons are non-interacting bosons (because of the linearity of the Maxwell equations)

The electromagnetic field operators are thus

$$\vec{A}_k(t) = \sqrt{\frac{2\pi\hbar c}{Vk}}\, e^{-i\omega_{\vec{k}} t}\, \vec{e}_k\, a_{\vec{k}}$$

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},\vec{e}_{\vec{k}}} \sqrt{\frac{2\pi\hbar c}{Vk}}\, \vec{e}_{\vec{k}} \left[e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} a_{\vec{k}} + e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} a^+_{\vec{k}} \right]$$

and correspondingly for $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$.

2° States of the radiation field

a) Photon number states and vacuum energy

A natural basis for the photon state space are the product states

$$|n_{\vec{k}_1}, n_{\vec{k}_2} \ldots \rangle = |n_{\vec{k}_1}\rangle |n_{\vec{k}_2}\rangle \ldots$$

where the mode with wave vector $\vec{k}_i$ contains $n_{\vec{k}_i}$ photons. These are energy eigenstates,

$$H\, |n_{\vec{k}_1} n_{\vec{k}_2} \ldots \rangle = \left(\sum_{\vec{k}} \hbar \omega_{\vec{k}} \left(n_{\vec{k}} + \frac{1}{2} \right) \right) |n_{\vec{k}_1} n_{\vec{k}_2} \ldots \rangle$$

and can be built up from the vacuum state $|0\rangle$ by successive application of the $a^{\dagger}_{\vec{k}}$ (Sect. II.2°)

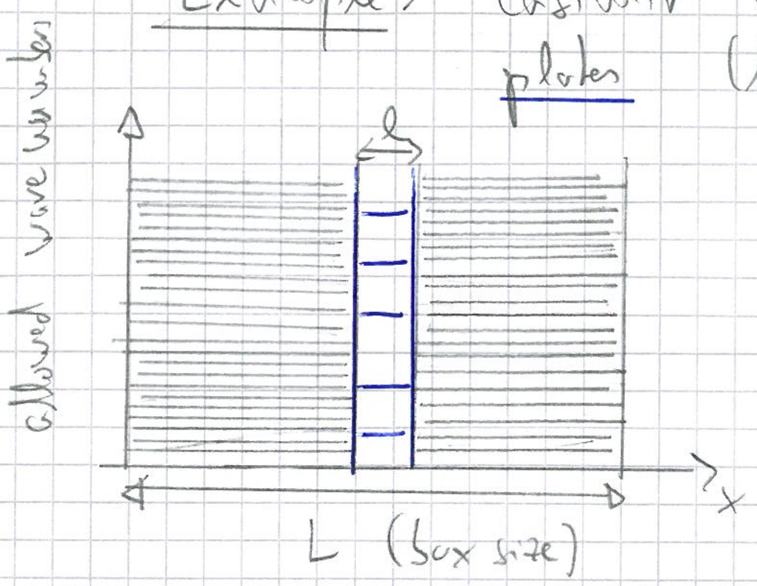
The energy of the vacuum (zero point energy)

$$E_0 = \frac{1}{2} \sum_{\vec{k}} \hbar \omega_{\vec{k}} \simeq \frac{1}{2} \left(\frac{V}{2\pi^2} \right) \int_0^{\infty} dk\, k^2\, \hbar c k = \infty$$

is infinite but generally unobservable.

However, since the sum over modes $\vec{k}$ depends on the shape and the boundary condition of the cavity to which the field is confined, differences in zero point energy can have observable effects.

Example: Casimir force between conducting plates (1948)



On the plates $\vec{E} = 0$
$\Rightarrow$ k_x is limited to values

$$k_x = \frac{\pi}{l} \nu'_x, \quad |\nu'_x| \geq 1$$

The smaller l, the more modes are excluded from the gap between the plates. This lowers the zero point energy and hence leads to a attractive force between the plates.

Magnitude of the force: We expect a constant force per area, which can depend on $\hbar c$ and l:

$$\frac{\text{Force}}{\text{Area}} = \frac{\text{Energy}}{\text{Volume}} \simeq \frac{\hbar c}{l^4}$$

The explicit calculation gives

$$\frac{F}{A} = -\frac{\pi^2}{240} \cdot \frac{\hbar c}{l^4}$$

An unambiguous experimental confirmation was achieved only in 1997.

b) Phase operators

We want to understand what the electromagnetic field of a single mode $\vec{k}$ with sharp photon number $n_{\vec{k}}$ looks like.

For a classical electric field mode

$$\vec{E}(\vec{r},t) = \vec{E}_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)} + \vec{E}^*_{\vec{k}}\, e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}$$

with $\vec{E}_{\vec{k}} = E_{\vec{k}}\, e^{i\phi_{\vec{k}}}\,\vec{e}_{\vec{k}}$

$E_{\vec{k}} \in \mathbb{R}$ amplitude, $\phi_{\vec{k}} \in \mathbb{R}$ phase

The corresponding field operator is

$$\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\sqrt{\frac{2\pi\hbar c}{Vk}}\,\vec{e}_{k}\Big(e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a_{\vec{k}} + e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a^{+}_{\vec{k}}\Big)$$

$$= i\,\vec{e}_{\vec{k}}\sqrt{\frac{2\pi\hbar ck}{V}}\Big(e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a_{\vec{k}} - e^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a^{+}_{\vec{k}}\Big)$$

Hence the quantization amounts to

$$E_{\vec{k}}\, e^{i\phi_{\vec{k}}} \longrightarrow i\sqrt{\frac{2\pi\hbar ck}{V}}\,a_{\vec{k}}$$

This suggest to decompose $a_{\vec{k}}$, $a^{+}_{\vec{k}}$ into "amplitude" and "phase" operators:

$$a_{\vec{k}} = \sqrt{n_{\vec{k}}+1}\, e^{i\phi_{\vec{k}}} \quad , \quad a^{+}_{\vec{k}} = e^{-i\phi_{\vec{k}}}\sqrt{n_{\vec{k}}+1}$$

where all quantities $(a_{\vec{k}}, n_{\vec{k}}, e^{i\phi_{\vec{k}}})$ designate operators.

In the following we drop the mode index $\vec{k}$ but add a $\hat{}$ to emphasize that we deal with operators:

$$\widehat{e^{i\phi}} = \sqrt{\hat{n}+1}^{\,-1}\,\hat{a} \quad , \quad \widehat{e^{-i\phi}} = \hat{a}^{+}\sqrt{\hat{n}+1}^{\,-1}$$

$$\Rightarrow \quad \widehat{e^{i\phi}}\,\widehat{e^{-i\phi}} = \sqrt{\hat{n}+1}^{\,-1}\,\underbrace{\hat{a}\,\hat{a}^{+}}_{\hat{n}+1}\sqrt{\hat{n}+1}^{\,-1} = 1$$

but note that $\widehat{e^{-i\phi}}\,\widehat{e^{i\phi}} \neq \widehat{e^{i\phi}}\,\widehat{e^{-i\phi}}$.

The action of the phase operators on photon number states are simple:

- $\widehat{e^{i\phi}}\,|n\rangle = \sqrt{\hat{n}+1}^{\,-1}\sqrt{n}\,|n-1\rangle = |n-1\rangle$ for $n>0$, and $\widehat{e^{i\phi}}\,|0\rangle = 0$
- $\widehat{e^{-i\phi}}\,|n\rangle = \hat{a}^{+}\sqrt{n+1}^{\,-1}\,|n\rangle = |n+1\rangle$

$$\Rightarrow \langle n-1|\,e^{i\phi}\,|n\rangle = \langle n+1|\,e^{-i\phi}\,|n\rangle = 1$$

and all other matrix elements vanish.

We can combine $e^{\widehat{i\phi}}$ and $e^{\widehat{-i\phi}}$ to form the Hermitian operators

$$\left.\begin{aligned}\widehat{\cos\phi} &= \frac{1}{2}\left(e^{\widehat{i\phi}} + e^{\widehat{-i\phi}}\right)\\ \widehat{\sin\phi} &= \frac{1}{2i}\left(e^{\widehat{i\phi}} - e^{\widehat{-i\phi}}\right)\end{aligned}\right\}$$

with matrix elements

$$\left.\begin{aligned}\langle n-1 \,|\, \cos\phi \,|\, n\rangle &= \langle n \,|\, \cos\phi \,|\, n-1\rangle = \frac{1}{2}\\ \langle n-1 \,|\, \sin\phi \,|\, n\rangle &= -\langle n \,|\, \sin\phi \,|\, n-1\rangle = \frac{1}{2i}\end{aligned}\right\}$$

The commutation relations (exercise)

$$[\hat{n}, \widehat{\cos\phi}] = -i\,\widehat{\sin\phi} \quad , \quad [\hat{n}, \widehat{\sin\phi}] = i\,\widehat{\cos\phi}$$

imply the general uncertainty relation

$$\left.\begin{aligned}\Delta n \cdot \Delta(\cos\phi) &\geq \frac{1}{2}|\langle \sin\phi \rangle|\\ \Delta n \cdot \Delta(\sin\phi) &\geq \frac{1}{2}|\langle \cos\phi \rangle|\end{aligned}\right\} \quad \text{see Sect. I. } 1^\circ$$

between photon number and phase.

Thus, these quantities cannot be simultaneously measured without uncertainty.

In the following we focus on the properties of photon number states.

In these states we have $\Delta n = 0$, and

$$\langle n | \cos\phi | n \rangle = \langle n | \sin\phi | n \rangle = 0$$

$$\langle n | \cos^2\phi | n \rangle = \langle n | \cos\phi \sum_m |m\rangle\langle m| \cos\phi | n \rangle$$

$$= \begin{cases} |\langle n | \cos\phi | n-1 \rangle|^2 + |\langle n | \cos\phi | n+1 \rangle|^2 = \frac{1}{2} & n > 0 \\ |\langle 0 | \cos\phi | 1 \rangle|^2 = \frac{1}{4} & n = 0 \end{cases}$$

Thus the phase uncertainty is, for $n > 0$,

$$\Delta(\cos\phi) = \Delta(\sin\phi) = \sqrt{\langle n | \cos^2\phi | n \rangle} = \frac{1}{\sqrt{2}}$$

Because the phase is defined on the bounded interval $[0, 2\pi]$, the phase uncertainty is maximal when ϕ is uniformly distributed:

$$\Delta(\cos\phi)_{max} = \left(\frac{1}{2\pi} \int_0^{2\pi} d\phi \, \cos^2\phi \right)^{1/2} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ photon number states with $n > 0$ have maximal phase uncertainty.