c) Coherent states

Exercise 6.1: Coherent states provide the "most classical" description of the harmonic oscillator.

Here we consider coherent states of the electromagnetic field, focusing on a single mode.

Definition: For a complex number $\alpha \in \mathbb{C}$ we define the superposition of photon number states

$$|\alpha\rangle := e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Properties: (i) $\langle\alpha|\alpha\rangle = 1$

(ii) $a|\alpha\rangle = \alpha|\alpha\rangle$

(iii) $\langle\alpha|\beta\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} \times$

$$\times \sum_{n=0}^{\infty} \frac{(\alpha^* \beta)^n}{n!} = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right]$$

$$\Rightarrow |\langle\alpha|\beta\rangle|^2 = \exp\left[-|\alpha-\beta|^2\right]$$

Coherent states are not orthogonal, but become approximately orthogonal for $|\alpha-\beta| \to \infty$.

They form a overcomplete basis of Hilbert space

In the following we explore some properties of these states.

(ii) Distribution of photon numbers:

The probability P_n to observe n photons is

$$P_n = |\langle\alpha|n\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

Poisson distribution with parameter $|\alpha|^2$

It follows from the properties of the Poisson distribution that

$$\langle\alpha|n|\alpha\rangle = |\alpha|^2$$

$$\langle\alpha|n^2|\alpha\rangle = |\alpha|^4 + |\alpha|^2 = \langle\alpha|n|\alpha\rangle^2 + \langle\alpha|n|\alpha\rangle$$

$$\Rightarrow (\Delta n)^2 = \langle\alpha|n^2|\alpha\rangle - \langle\alpha|n|\alpha\rangle^2 = \langle\alpha|n|\alpha\rangle = |\alpha|^2$$

The variance of the Poisson distribution is equal to the mean

The relative fluctuation of the photon number is thus (coefficient of variation)

$$\frac{\Delta n}{\langle n\rangle} = \frac{1}{|\alpha|} \to 0 \quad \text{for } |\alpha| \to \infty$$

On the other hand the index of dispersion is defined as

$$\frac{(\Delta n)^2}{\langle n\rangle} = 1 \quad \text{for the Poisson distribution}$$

(ii) Phase fluctuations: With $\alpha = |\alpha| e^{i\theta}$

$$\langle \alpha | \hat{\cos\phi} | \alpha \rangle = e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^n (\alpha^*)^m}{(n!\, m!)^{1/2}} \underbrace{\langle m | \hat{\cos\phi} | n \rangle}_{= \frac{1}{2} (\delta_{m\, n+1} + \delta_{m\, n-1})}$$

$$= \frac{1}{2} e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha^*)^{n+1} \alpha^n + \alpha^{n+1} (\alpha^*)^n}{((n+1)!\, n!)^{1/2}} =$$

$$= e^{-|\alpha|^2} \underbrace{\frac{1}{2} (\alpha^* + \alpha)}_{|\alpha| \cos\theta} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n! \sqrt{n+1}} =$$

$$= \cos\theta \; \underbrace{|\alpha| e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n! \sqrt{n+1}}}_{\to \simeq 1 - \frac{1}{8|\alpha|^2} \text{ for } |\alpha| \to \infty}$$

(Carruthers, Nieto 1965)

Similarly it can be shown that

$$\langle \alpha | \hat{\cos}^2\phi | \alpha \rangle \simeq \cos^2\theta - \frac{\cos^2\theta - 1/2}{2|\alpha|^2} + \ldots$$

for large $|\alpha|$. This implies

$$[\Delta(\cos\phi)]^2 = \frac{1 - \cos^2\theta}{4|\alpha|^2} = \frac{\sin^2\theta}{4|\alpha|^2}$$

$$\Rightarrow \quad \Delta n \cdot \Delta \cos\varphi = \frac{1}{2}|\sin\Theta|$$

which saturates the uncertainty relation. Thus the coherent states "realize" the smallest possible uncertainty product for phase and photon number, similar to Gaussian wavepackets in elementary quantum mechanics.

(iii) Electric field: The expectation of a single field mode with wave vector $\vec{k}$ is

$$\langle\alpha|\vec{E}(\vec{r},t)|\alpha\rangle = i\,\vec{e}_k\sqrt{\frac{2\pi\hbar c k}{V}} \times$$

$$\times\Big(e^{i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\langle\alpha|a|\alpha\rangle}_{\alpha} - e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\langle\alpha|a^\dagger|\alpha\rangle}_{\alpha^*}\Big)$$

$$= -2\,\vec{e}_k\sqrt{\frac{2\pi\hbar c k}{V}}\,|\alpha|\sin(\vec{k}\cdot\vec{r}-\omega t+\Theta)$$

To compute the uncertainty we also need

$$\langle\alpha||\vec{E}(\vec{r},t)|^2|\alpha\rangle = -\frac{2\pi\hbar c k}{V}\Big(e^{2i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\langle\alpha|a^2|\alpha\rangle}_{=\alpha^2}$$

$$+\,e^{-2i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\langle\alpha|(a^\dagger)^2|\alpha\rangle}_{=(\alpha^*)^2} - \underbrace{\langle\alpha|a^\dagger a|\alpha\rangle}_{=|\alpha|^2} -$$

$$-\,\underbrace{\langle\alpha|a a^\dagger|\alpha\rangle}\Big) \rightarrow\; = 1+|\alpha|^2$$

$$= \langle\alpha|\vec{E}(\vec{r},t)|\alpha\rangle^2 + \frac{2\pi\hbar c k}{V}$$

$$\Rightarrow \quad \Delta\vec{E} = \sqrt{\frac{2\pi\hbar c k}{V}} \quad \text{independent of } \alpha$$

It follows that the relative uncertainty of the field strength is

$$\frac{\Delta\vec{E}}{|\langle\alpha|\vec{E}|\alpha\rangle|} = \frac{1}{2|\alpha|} \to 0 \;, \quad |\alpha| \to \infty$$

In the limit $|\alpha| \to \infty$ the coherent state converges to a classical field with phase θ and amplitude

$$|\vec{E}| = \sqrt{\frac{8\pi\hbar c k}{V}}\,|\alpha|$$

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d) Mixed states

Most real light sources produce states that are mixed and thus need to be described by a density operator ρ.

Example: In a thermal radiation field at temperature T, the probability for a field mode to be in a state with n photons is

$$p_n = \frac{1}{Z} e^{-\beta E_n} \quad , \quad E_n = \hbar\omega(n+\tfrac{1}{2})$$

with the canonical partition function

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\Rightarrow \quad p_n = (1-e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega n} \qquad \beta = \frac{1}{k_B T}$$

The corresponding density operator is

$$\rho = \sum_{n=0}^{\infty} p_n |n\rangle\langle n| = (1-e^{-\beta\hbar\omega}) \sum_n e^{-n\beta\hbar\omega} |n\rangle\langle n|$$

The mean number of photons $\bar{n}$ is

$$\bar{n} = \mathrm{Tr}(\rho\, a^+ a) = (1-e^{-\beta\hbar\omega}) \sum_{n=0}^{\infty} n (e^{-\beta\hbar\omega})^n$$

$$= \frac{1}{e^{\beta\hbar\omega}-1}$$ Bose-Einstein distribution with $\mu=0$

It follows that $e^{-\beta\hbar\omega} = \frac{\bar{n}}{\bar{n}+1}$, which allows to express ρ as

$$\rho = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} |n\rangle\langle n|$$

$*$

The distribution of photon numbers is geometric.

Density operators of the form ⊛ appear also beyond thermal equilibrium as a description of light fields that are maximally random ("chaotic").

To quantify this notion we introduce the Shannon entropy of a probability distribution as

$$S[\{p_n\}] = -\sum_n p_n \ln p_n$$

which we want to maximize subject to the constraints

$$\sum_{n=0}^{\infty} p_n = 1 \quad , \quad \sum_{n=0}^{\infty} n p_n = \bar{n}$$

This yields the conditions

$$\frac{d}{dp_k}\left(-\sum_n p_n \ln p_n - \lambda_1 \sum_n n p_n - \lambda_2 \sum_n p_n\right) = 0$$

$$\Rightarrow -\ln p_k - 1 - k\lambda_1 - \lambda_2 = 0$$

$$\Rightarrow p_k = e^{-(\lambda_1 k + \lambda_2 + 1)} = e^{-\lambda_2 + 1} (\underbrace{e^{-\lambda_1}}_{=:b})^k$$

$$= (1-b)\, b^k \quad \text{geometric distribution}$$

with $\bar{n} = \frac{b}{1-b}$ $\Rightarrow$ $p_k = \frac{(\bar{n})^k}{(\bar{n}+1)^{k+1}}$

The general density operator for a chaotic radiation field with multiple modes is thus

$$\rho = \sum_{n_{\vec{k}_1}} \sum_{n_{\vec{k}_2}} \dots P(n_{\vec{k}_1}, n_{\vec{k}_2} \dots) \, | n_{\vec{k}_1}, n_{\vec{k}_2} \dots \rangle \langle n_{\vec{k}_1}, n_{\vec{k}_2} \dots |$$

with $P(n_{\vec{k}_1}, n_{\vec{k}_2} \dots) := \prod_{\vec{k}_i} \frac{(\bar{n}_{\vec{k}_i})^{n_{\vec{k}_i}}}{(\bar{n}_{\vec{k}_i}+1)^{n_{\vec{k}_i}+1}}$.

where $\bar{n}_{\vec{k}_i}$ is the mean number of photons with wave number $\vec{k}_i$ (not necessarily thermal)

Density operator can be represented in terms of coherent states using Glauber's P-representation

$$\rho = \int_{\mathbb{C}} d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle\alpha|$$

where $P(\alpha)$ is a real-valued function and the element of integration is $d^2\alpha = d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha)$ i.e. $\mathbb{C}$ is treated as $\mathbb{R}^2$.

The requirements on $P(\alpha)$ are

(i) $\mathrm{Tr}(\rho) = \sum_{n=0}^{\infty} \langle n | \rho | n \rangle = \int d^2\alpha \, P(\alpha) \underbrace{\sum_{n=0}^{\infty} |\langle n | \alpha \rangle|^2}_{=1}$

$\overset{!}{=} 1$

$\Rightarrow$ $P(\alpha)$ is normalized.

(ii) $\langle\alpha|\rho|\alpha\rangle = \int d^2\beta \, P(\beta) \, |\langle\alpha|\beta\rangle|^2$

$$= \int d^2\beta \, P(\beta) \, e^{-|\alpha-\beta|^2} \geq 0$$

which does not require $P(\beta) \geq 0$ everywhere

For a chaotic light field we have

$$\langle\alpha|\rho|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \cdot \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} =$$

$$= \frac{e^{-|\alpha|^2}}{(1+\bar{n})} \exp\left[|\alpha|^2 \frac{\bar{n}}{1+\bar{n}}\right] =$$

$$= \frac{1}{1+\bar{n}} \exp\left[-\frac{|\alpha|^2}{1+\bar{n}}\right]$$

from which it can be deduced that

$$P(\alpha) = \frac{1}{\pi\bar{n}} e^{-|\alpha|^2/\bar{n}}$$ exercise

3° Light - matter interaction

We want to describe the interaction of the quantized radiation field with an atom.

The usual semiclassical theory starts from the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}(\vec{r},t)\right)^2 + U_c(\vec{r}).$$

where $\vec{p}$ is the momentum operator, $U_c(\vec{r})$ is the Coulomb potential and $\vec{A}(\vec{r},t)$ a classical vector potential.

For weak radiation fields H can be linearized in $\vec{A}$:

$$H \simeq \underbrace{\frac{1}{2m}\vec{p}^{\,2} + U_c(\vec{r})}_{= H_0 \text{ atomic Hamiltonian}} - \frac{e}{2cm}\left(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}\right)$$

With $\vec{p} = -i\hbar\nabla$ we see that

$$\vec{p}\cdot\vec{A} = \vec{A}\cdot\vec{p} - i\hbar\underbrace{\nabla\cdot\vec{A}}_{=0 \text{ in Coulomb gauge}} = \vec{A}\cdot\vec{p}$$

$$\Rightarrow \quad H = H_0 + \frac{ie\hbar}{mc}\vec{A}(\vec{r},t)\cdot\nabla$$

We consider transitions between states

$$|i\rangle = |\nu, n\rangle \quad \text{and} \quad |f\rangle = |\nu', n'\rangle$$

atomic state

photon number

According to Fermi's golden rule the transition rate is

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\frac{e^2}{m c^2}|\langle f|\vec{A}\cdot\vec{p}|i\rangle|^2\,\delta(E_f - E_i)$$

where the matrix element is to be evaluated for the operator of a single mode of the radiation field (in the Schrödinger picture)

$$\vec{A}(\vec{r}) = \sqrt{\frac{2\pi\hbar c}{Vk}}\,\vec{e}_{\vec{k}}\left(e^{i\vec{k}\cdot\vec{r}}a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{r}}a^{+}_{\vec{k}}\right)$$

Two cases:

(i) Emission of a photon ($n' = n+1$)

In this case only $a^{+}_{\vec{k}}$ contributes

$$\Rightarrow \langle f|\vec{A}\cdot\vec{p}|i\rangle = \sqrt{n+1}\sqrt{\frac{2\pi\hbar c}{Vk}}\langle \nu'|e^{-i\vec{k}\cdot\vec{r}}\,\vec{e}_{k}\cdot\vec{p}|\nu\rangle$$

Dipole approximation: $e^{-i\vec{k}\cdot\vec{r}} \simeq 1$ if the wavelength of light is large compared to the size of the atom

Moreover we use the relation

$$\vec{p} = -\frac{im}{\hbar}[\vec{r}, H_0]$$

which can be proved using exercise 1.1

$$\Rightarrow \langle \nu | \vec{e}_{\vec{k}} \cdot \vec{p} | \nu' \rangle = -\frac{im}{\hbar} \langle \nu' | \vec{e}_{\vec{k}} \cdot \vec{r} H_0 - H_0 \vec{e}_{\vec{k}} \cdot \vec{r} | \nu \rangle$$

$$= -\frac{im}{\hbar} \underbrace{(E_\nu - E_{\nu'})}_{= \hbar\omega_{\vec{k}} \text{ by energy conservation}} \underbrace{\langle \nu' | \vec{e}_{\vec{k}} \cdot \vec{r} | \nu \rangle}_{=: \vec{e}_{\vec{k}} \cdot \vec{r}_{\nu\nu'}} \quad \text{dipole matrix element}$$

It follows that

$$\Gamma^{(em)}_{i \to f} = \frac{4\pi^2 e^2 \omega_k}{V} (n+1) |\vec{e}_{\vec{k}} \cdot \vec{r}_{\nu\nu'}|^2 \delta(E_\nu - E_{\nu'} - \hbar\omega_{\vec{k}})$$

For $n = 0$ this describes the rate of spontaneous emission, which is a consequence of the zero point fluctuations of the electromagnetic field and was predicted by Einstein (1916/17) using the condition of detailed balance with respect to Planck's radiation law.

(ii) Absorption of a photon $(n' = n-1)$

In this case only $a_{\vec{k}}$ contributes and an analogous calculation yields $(n > 0)$

$$\Gamma_{i\to f}^{abs} = \frac{4\pi^2 e^2 \omega_{\vec{k}}}{V} \, n \, |\vec{e}_{\vec{k}} \cdot \vec{r}_{\nu\nu'})|^2 \, \delta(E_\nu - E_{\nu'} + \hbar\omega_{\vec{k}})$$

This suggests more generally to decompose $\vec{A}$ or $\vec{E}$ into two parts describing absorption and emission of photons, respectively:

$$\vec{E}(\vec{r},t) = \Big(\underbrace{\sum_{\vec{k}} i \sqrt{\frac{2\pi\hbar\omega_{\vec{k}}}{V}} \, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \, a_{\vec{k}}}_{=: E^{(+)}(\vec{r},t)} +$$

$$+ \underbrace{\sum_{\vec{k}} (-i) \sqrt{\frac{2\pi\hbar\omega_{\vec{k}}}{V}} \, e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} \, a_{\vec{k}}^{+}}_{=: E^{(-)}(\vec{r},t)} \Big) \, \vec{e}$$

where a single polarization vector $\vec{e}$ has been assumed for convenience.

$E^{(+)}$ and $E^{(-)}$ are analogous to the field operators ψ, ψ^+ from Chapter II and satisfy

$$\underline{(E^{(+)})^+ = E^{(-)}}$$

The rate of absorption of photons is then of the general form

$$\Gamma^{(abs)}_{i \to f} = |M|^2 \, |\langle f | E^{(+)}(\vec{r},t) | i \rangle|^2$$

where M is some generalized matrix element[1) describing the "detector"] and $|i\rangle$, $|f\rangle$ are the initial and final state of the radiation field.

If the initial state is mixed with density operator

$$\rho = \sum_i p_i |i\rangle\langle i|$$

the total absorption rate is

$$\Gamma^{(abs)} = \sum_i p_i \, |M|^2 \underbrace{\sum_f |\langle f | E^{(+)}(\vec{r},t) | i \rangle|^2}$$

$$= \sum_f \langle i | E^{(-)}(\vec{r},t) | f \rangle \langle f | E^{(+)}(\vec{r},t) | i \rangle =$$

$$= \langle i | E^{(-)}(\vec{r},t) \, E^{(+)}(\vec{r},t) | i \rangle$$

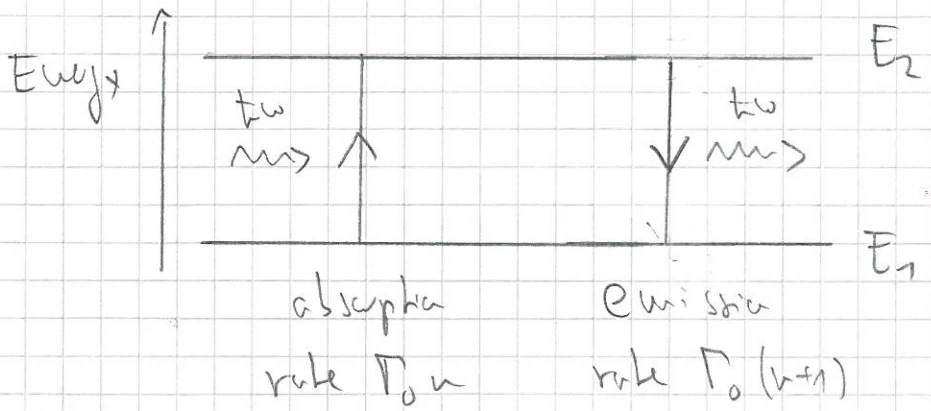
$$\Rightarrow \Gamma^{(abs)} = |M|^2 \sum_i p_i \langle i | E^{(-)}(\vec{r},t) \, E^{(+)}(\vec{r},t) | i \rangle$$

$$= |M|^2 \, \mathrm{Tr}(\rho \, E^{(-)} E^{(+)}) = |M|^2 \, \mathrm{Tr}(\rho \, \hat{I})$$

where $\hat{I}(\vec{r},t) = E^{(-)}(\vec{r},t) \, E^{(+)}(\vec{r},t)$ is the operator describing the field intensity.

A simple model for the generation of light

We consider the interaction of a single light mode with a collection of 2-level atoms:



N_1 atoms in $|1\rangle$

N_2 atoms in $|2\rangle$

n photons

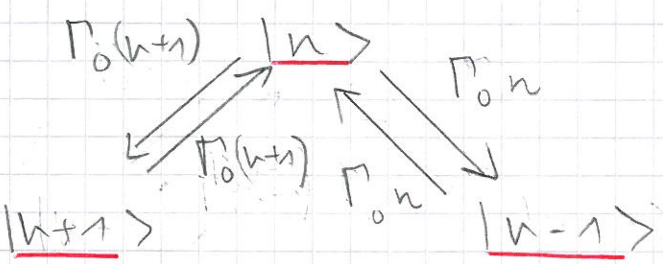
Γ_0: rate of spontaneous emission

For simplicity we assume that N_1 and N_2 are kept constant, and ask for the time evolution of the probability distribution of the number of photons $p_n(t)$.

This is governed by a master equation that accounts for the gain and loss of probability of the different photon states:

$|n\rangle$ gains by absorption from state $|n+1\rangle$ and emission from state $|n-1\rangle$

$|n\rangle$ loses by absorption and emission to states $|n-1\rangle$ and $|n+1\rangle$:



$$\Rightarrow \quad \frac{d}{dt} p_n = \underbrace{\Gamma_0 \, n \, N_2 \, p_{n-1} + \Gamma_0 (n+1) N_1 \, p_{n+1}}_{\text{gain}}$$

$$- \underbrace{\left(\Gamma_0 \, n \, N_1 + \Gamma_0 (n+1) N_2 \right)}_{\text{loss}} p_n$$

$n \geq 1$

For $n = 0$ two of the terms disappear:

$$\frac{d}{dt} p_0 = \Gamma_0 N_1 p_1 - \Gamma_0 N_2 p_0 .$$

In the exercises solutions to these equations will be studied.