With the product ansatz $\psi(\vec{r}) = \varphi_\ell(r) Y_\ell^m(\theta, \phi)$ this gives the radial Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2}\right) + V(r)\right]\varphi_\ell(r) = E\varphi_\ell(r)$$

To simplify the radial derivative we define

$$\underline{\varphi_\ell(r) = \frac{1}{r} u_\ell(r)}$$

then the free particle with energy $E = \frac{\hbar^2}{2m} k^2$ satisfies

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2\right) u_\ell(r) = 0$$

With the dimensionless spatial coordinate $\rho = rk$ this finally yields

$$\underline{\left(\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + 1\right) u_\ell(\rho) = 0}$$

We analyze the behavior of solutions for $\rho \to 0$ and $\rho \to \infty$, respectively.

(i) $\underline{\rho \to 0:}$ $\quad \frac{d^2}{d\rho^2} u_\ell = \frac{\ell(\ell+1)}{\rho^2} u_\ell$

$\underline{\text{Ansatz:}}$ $\quad u_\ell = \rho^\alpha \Rightarrow u_\ell'' = \alpha(\alpha-1)\rho^{\alpha-2} =$

$$= \frac{\alpha(\alpha-1)}{\rho^2} u_\ell \quad \Rightarrow \quad \alpha(\alpha-1) = \ell(\ell+1)$$

This quadratic equation has two solutions:

$$\left.\begin{array}{l} \alpha = \ell+1 \Rightarrow \text{regular solution} \quad u_\ell \sim \rho^{\ell+1} \\ \alpha = -\ell \Rightarrow \text{irregular solution} \quad u_\ell \sim \rho^{-\ell} \end{array}\right\}$$

(ii) $\rho \to \infty$:

$$\left.\begin{array}{l} \frac{d^2}{d\rho^2} u_\ell = -u_\ell \\ \Rightarrow u_\ell(\rho) = A \sin\rho + B\cos\rho \end{array}\right\}$$

The full solutions are given in terms of spherical Bessel functions j_ℓ, y_ℓ as follows:

- regular solutions:
$$\left.\begin{array}{l} u_\ell = \rho\, j_\ell(\rho) \\ j_\ell(\rho) = \rho^\ell \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\sin\rho}{\rho} \end{array}\right\}$$

- irregular solutions:

Neumann fcts.

$$\left.\begin{array}{l} u_\ell(\rho) = \rho\, y_\ell(\rho) \\ y_\ell(\rho) = -\rho^\ell \left(-\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \frac{\cos\rho}{\rho} \end{array}\right\}$$

To identify the contribution to the asymptotic $\mathcal{S}$ we need to investigate the behavior of the exact solution for $\rho \to \infty$.

In the regular case we have

$$u_0(\rho) = \sin\rho$$

$$u_1(\rho) = \rho \left\{ \rho \left(-\frac{1}{\rho}\frac{d}{d\rho} \right) \frac{\sin\rho}{\rho} \right\} =$$

$$= \rho \left\{ \frac{1}{\rho^2}\sin\rho - \frac{1}{\rho}\cos\rho \right\} =$$

$$= \frac{1}{\rho}\sin\rho - \cos\rho$$

$$u_2(\rho) = \underbrace{\frac{2}{\rho}\left(\frac{1}{\rho} - \cos\rho\right) - \frac{1}{\rho^2}\sin\rho - \frac{1}{\rho}\cos\rho}_{\to 0,\ \rho \to \infty} - \sin\rho$$

In general, it can be seen from the definition of j_ℓ and y_ℓ that there is always one term where all derivatives act on $\sin\rho$ and $\cos\rho$, respectively, and powers of ρ cancel:

$$\left.\begin{aligned} j_\ell(\rho) &\to \frac{1}{\rho}(-1)^\ell \left(\frac{d}{d\rho}\right)^\ell \sin\rho \\ y_\ell(\rho) &\to \frac{1}{\rho}(-1)^{\ell+1} \left(\frac{d}{d\rho}\right)^\ell \cos\rho \end{aligned}\right\} \rho \to \infty$$

Each derivative applied to a trigonometric function shifts the phase by $\pi/2$:

$$\left.\begin{aligned} \left(\frac{d}{d\rho}\right)^l \sin\rho &= \sin\left(\rho + \frac{l\pi}{2}\right) \\ \left(\frac{d}{d\rho}\right)^l \cos\rho &= \cos\left(\rho + \frac{l\pi}{2}\right) \end{aligned}\right\}$$

Thus the Hankel functions $h_l^{(1,2)}(\rho)$ defined by

$$\left.\begin{aligned} h_l^{(1)}(\rho) &= j_l(\rho) + i y_l(\rho) \\ h_l^{(2)}(\rho) &= j_l(\rho) - i y_l(\rho) \end{aligned}\right\}$$ have the asymptotic behavior

$$h_l^{(1)}(\rho) \simeq \frac{1}{\rho} \underbrace{(-1)^l}_{(e^{i\pi})^l} \Big(\underbrace{\sin\left(\rho + \frac{l\pi}{2}\right) - i\cos\left(\rho + \frac{l\pi}{2}\right)}_{= \frac{1}{i} e^{i\left(\rho + \frac{l\pi}{2}\right)}} \Big)$$

$$= \frac{1}{\rho i} e^{i\left(\rho - \frac{l\pi}{2}\right)}$$

Thus these solutions have the form of a outgoing spherical wave

Similarly $h_\ell^{(2)}(\rho) \simeq \frac{i}{\rho}\, e^{-i(\rho - \frac{\ell\pi}{2})}$.

---

b) Scattering phases

In the following we assume that the potential is of strictly finite range, i.e

$$V(r) = 0 \quad \text{for} \quad r > R$$

Then the total wave function

$$\psi(\vec{r}) = e^{ikz} + \psi_s(\vec{r}) \qquad (A=1)$$

is a solution of the free particle Schrödinger equation with asymptotics (S), i.e.

(S') $$\psi_s(\vec{r}) \simeq f(\theta,\phi)\,\frac{e^{ikr}}{r}, \quad r \to \infty$$

The most general expansion of ψ_s is

$$\psi_s(\vec{r}) = \sum_{\ell,m} \varphi_\ell(r)\, Y_\ell^m(\theta,\phi)$$

However, because of the radial symmetry of the potential there cannot be any dependence on ϕ, i.e. only terms with $m=0$ contribute:

$$\psi_s(\vec{r}) = \sum_\ell \phi_\ell(r)\, Y_\ell^0(\theta) = \sum_\ell \phi_\ell(r) \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

where $P_\ell(x) = 2^{-\ell}(\ell!)^{-1}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$

are Legendre polynomials defined on $(-1, 1)$.

To satisfy the boundary condition S^i the ϕ_ℓ have to be proportional to $h_\ell^{(1)}$:

$$\psi_s(\vec{r}) = \sum_\ell i^\ell \left(\frac{2\ell+2}{2}\right) a_\ell\, h_\ell^{(1)}(kr)\, P_\ell(\cos\theta)$$

For $|\vec{r}| \to \infty$ $\quad h_\ell^{(1)} \simeq \frac{1}{ikr}\, e^{i\left(kr - \frac{\ell\pi}{2}\right)}$

$$= \underbrace{\left(\frac{1}{i}\, e^{-i\frac{\pi}{2}\ell}\right)}_{=\left(\frac{1}{i}\right)^{\ell+1}} \frac{e^{ikr}}{kr}$$

$$\Rightarrow \psi_s(\vec{r}) \xrightarrow[r\to\infty]{} \underbrace{\left[\frac{1}{2ik}\sum_\ell (2\ell+1)\, a_\ell\, P_\ell(\cos\theta)\right]}_{=f(\theta)} \frac{e^{ikr}}{r}$$

The form of the expansion coefficients becomes clear by considering the representation of the incoming plane wave in terms of spherical harmonics:

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l\, P_l(\cos\theta)\, j_l(kr)$$

Rayleigh formula

With this the full wave function becomes

$$\psi(\vec{r}) = e^{ikz} + \psi_s(\vec{r}) =$$

$$= \sum_{l=0}^{\infty} (2l+1)\, i^l\, P_l(\cos\theta)\left[j_l(kr) + \frac{a_l}{2} h_l^{(1)}(kr) \right]$$

For $r \to \infty$

$$j_l(kr) + \frac{a_l}{2} h_l^{(1)}(kr) \longrightarrow \frac{1}{kr}\left[\overbrace{\sin\left(kr - \frac{\pi l}{2}\right)}^{(-1)^l \sin\left(kr + \frac{\pi l}{2}\right)} + \frac{a_l}{2i} e^{i\left(kr - \frac{\pi l}{2}\right)} \right]$$

$$= \frac{1}{2ikr}\left[e^{i\left(kr - \frac{\pi l}{2}\right)} - e^{-i\left(kr - \frac{\pi l}{2}\right)} + a_l\, e^{i\left(kr - \frac{\pi l}{2}\right)} \right] =$$

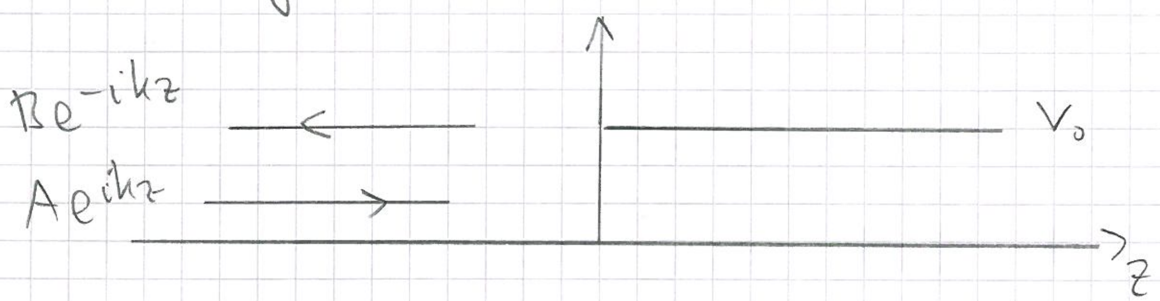
$$= \frac{i}{2kr}\left[\underbrace{e^{-i\left(kr - \frac{\pi l}{2}\right)}}_{\text{incoming}} - \underbrace{\overbrace{(1+a_l)}^{=: S_l}\, e^{i\left(kr - \frac{\pi l}{2}\right)}}_{\text{outgoing spherical wave}} \right]$$

S_ℓ is called the "scattering matrix", which is diagonal in this case. Because of probability conservation $|S_\ell|^2 = 1$, which implies that

$$S_\ell = e^{2i\delta_\ell}$$

with the scattering phases $\delta_\ell \in [-\pi/2, \pi/2]$.

Analogy to one-dimensional scattering:



then for $E < V_0$ $|A|^2 = |B|^2$, but the reflected wave function has a different phase:

$$B = e^{2i\delta} A$$

The terms in the decomposition of $\psi_s(\vec{r})$ are called partial waves, and the scattering phase δ_ℓ completely characterizes the scattering of the ℓ-th partial wave.

c) Scattering amplitude and cross section

The scattering amplitude is given in terms of the scattering phases as

$$\underline{f(\theta)} = \frac{1}{2ik} \sum_l (2l+1)\left(e^{2i\delta_l} - 1\right) P_l(\cos\theta) =$$

$$= \underline{\frac{1}{k} \sum_l (2l+1) e^{i\delta_l} \sin(\delta_l) P_l(\cos\theta)}$$

and the differential cross section is $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$.

Thus the total cross section is

$$\sigma_{tot} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, |f(\theta)|^2 =$$

$$= \frac{2\pi}{k^2} \sum_l \sum_{l'} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin(\delta_l) \sin(\delta_{l'}) \times$$

$$\times \underbrace{\int_0^{\pi} d\theta \, \sin\theta \, P_l(\cos\theta) P_{l'}(\cos\theta)}$$

$$= \int_{-1}^{1} d(\cos\theta) \, P_l(\cos\theta) P_{l'}(\cos\theta) =$$

$$= \frac{2}{2l+1} \delta_{ll'} \quad \text{orthogonality relation}$$

$$\Rightarrow \sigma_{tot} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

On the other hand, because $P_l(1) = 1 \; \forall l$,

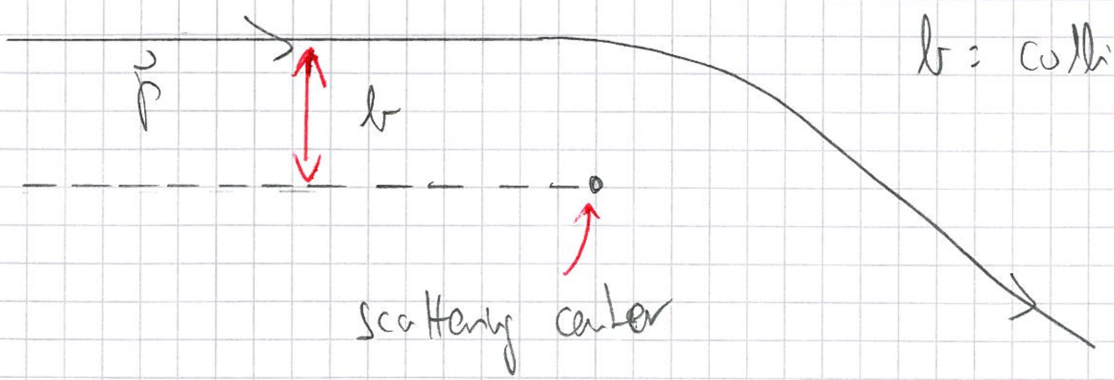
$$f(0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l =$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \left(\cos(\delta_l) \sin(\delta_l) + i \sin^2\delta_l \right)$$

$$\Rightarrow \operatorname{Im} f(0) = \frac{k}{4\pi} \sigma_{tot}$$ optical theorem

which is a consequence of probability conservation (exercise 13.1)

To obtain a general bound on σ_{tot} we invoke a classical picture of the scattering process:



b = collision parameter

$\Rightarrow$ angular momentum of the particle is $\hbar l = bp$

$$\Rightarrow \quad l = \frac{bp}{\hbar} = \underline{bk} \quad \Rightarrow \quad b = l/k$$

If the range of the potential is R, then no scattering occurs for $b > R$

$$\Rightarrow \quad \underline{l \leq l_{max} = kR}$$ contribute to the sum over l.

$$\Rightarrow \underline{\sigma_{tot}} = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l \leq$$

$$\leq \frac{4\pi}{k^2} \sum_{l=0}^{l_{max}} (2l+1) = \frac{4\pi}{k^2} \left(2 \cdot \frac{l_{max}(l_{max}+1)}{2} + l_{max} \right)$$

$$= \frac{4\pi}{k^2} \left(l_{max}^2 + 2 l_{max} \right) \simeq \underline{4\pi R^2 = 4\, \sigma_{geom}}$$

for $l_{max} \gg 1$, where $\sigma_{geom} = \pi R^2$ is the expected geometric cross section.

5° Hard sphere scattering

For the hard sphere potential

$$V(r) = \begin{cases} 0 & r > R \\ \infty & r \leq R \end{cases}$$

The radial wave functions have to vanish at $r = R$:

$$\underbrace{j_\ell(kR) + \frac{a_\ell}{2} h_\ell^{(1)}(kR)}_{} = 0 \qquad \forall \ell$$

$$= \frac{1}{2}\left\{ h_\ell^{(1)}(kR) + h_\ell^{(2)}(kR) + a_\ell h_\ell^{(1)}(kR) \right\}$$

$$= \frac{1}{2}\left\{ \underbrace{(1 + a_\ell)}_{= e^{2i\delta_\ell}} h_\ell^{(1)}(kR) + h_\ell^{(2)}(kR) \right\} =$$

$$= \frac{1}{2} e^{i\delta_\ell} \left\{ \underbrace{e^{i\delta_\ell} h_\ell^{(1)}(kR) + e^{-i\delta_\ell} h_\ell^{(2)}(kR)}_{} \right\}$$

$$= 2\,\mathrm{Re}\left(h_\ell^{(1)}(kR)\, e^{i\delta_\ell} \right) =$$

$$= \underline{2 \left\{ j_\ell(kR)\cos(\delta_\ell) - y_\ell(kR)\sin(\delta_\ell) \right\}}$$

because $\left(h_\ell^{(2)}\right)^* = h_\ell^{(1)}$. Thus the scattering phases are given by

$$\underline{\tan(\delta_\ell) = \frac{j_\ell(kR)}{y_\ell(kR)}}$$

a) Low energy scattering

Recall that for $\rho \to 0$ $\quad j_l(\rho) \sim \rho^l$, $y_l(\rho) \sim \rho^{-(l+1)}$

$$\Rightarrow \frac{j_l(kR)}{y_l(kR)} \sim (kR)^{2l+1} \quad , \quad kR \to 0$$

The dominant contribution to the scattering cross section comes from $l=0$, where

$$\frac{j_0(kR)}{y_0(kR)} = -\tan(kR) \quad \Rightarrow \quad \delta_0 \simeq -kR$$

This motivates to introduce the scattering length

$$a = -\lim_{k \to 0} \frac{\delta_0(k)}{k}$$

which in this case is equal to R.

Taking into account only the $l=0$ term, the scattering amplitude is

$$f(\theta) = \frac{1}{k} e^{i\delta_0} \sin(\delta_0) \underbrace{P_0(\cos\theta)}_{=1} = \frac{1}{k} e^{i\delta_0} \sin(\delta_0)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f|^2 = \frac{1}{k^2} \sin^2(\delta_0) \approx \frac{\delta_0^2}{k^2} = R^2$$

Scattering is isotropic, and the total

cross section is

$$\sigma_{tot} = 4\pi \frac{d\sigma}{d\Omega} = 4\pi R^2$$

Also note that

$$\mathrm{Im}(f) = \frac{1}{k} \sin^2(\delta_0) = \frac{k}{4\pi} \sigma_{tot}$$

as required by the optical theorem

---

b) High energy scattering

As discussed above, the total cross section is

$$\sigma_{tot} \simeq \frac{4\pi}{k^2} \sum_{l=0}^{l_{max}} (2l+1) \sin^2(\delta_l) \quad \text{with } l_{max} = kR \gg 1$$

For the hard sphere

$$\sin^2(\delta_l) = \frac{\sin^2(\delta_l)}{\sin^2(\delta_l) + \cos^2(\delta_l)} = \frac{\tan^2(\delta_l)}{1 + \tan^2(\delta_l)} =$$

$$= \frac{(j_l / y_l)^2}{1 + (j_l / y_l)^2} = \frac{j_l^2(kR)}{j_l^2(kR) + y_l^2(kR)}$$

Moreover

$$\left.\begin{aligned} j_l(\rho) &\simeq (-1)^l \frac{1}{\rho} \sin\left(\rho - \frac{l\pi}{2}\right) \\ y_l(\rho) &\simeq (-1)^{l+1} \frac{1}{\rho} \cos\left(\rho - \frac{l\pi}{2}\right) \end{aligned}\right\} \text{ for } \rho \to \infty$$

$$\Rightarrow \quad \sin^2(\delta_l) \cong \sin^2\left(kR - \frac{l\pi}{2}\right)$$

This implies that

$$\sin^2(\delta_l) + \sin^2(\delta_{l+1}) = \sin^2\left(kR - \frac{l\pi}{2}\right) + \underbrace{+\sin^2\left(kR - \frac{l\pi}{2} - \frac{\pi}{2}\right)}_{= \cos^2\left(kR - \frac{l\pi}{2}\right)} = 1$$

independent of l. Thus $\sin^2(\delta_l)$ can be replaced by $\frac{1}{2}$ in the sum and

$$\sigma_{tot} \simeq \frac{2\pi}{k^2} \sum_{l=0}^{l_{max}} (2l+1) = \frac{2\pi}{k^2}\left(l_{max}^2 + 2l_{max}\right)$$

$$\simeq 2\pi R^2 \quad \text{for } kR \gg 1.$$

$$= 2\,\sigma_{geom}.$$

On the other hand the limit

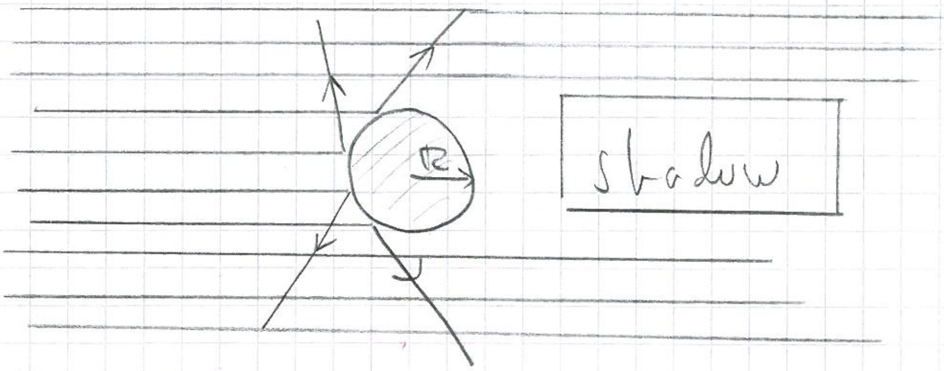
$$kR = \frac{p}{\hbar} R \to \infty$$

can also be understood as a classical limit ($\hbar \to 0$) at fixed momentum p, and therefore one would expect

$$\sigma_{tot} \to \sigma_{geom} = \pi R^2.$$

Where does the factor 2 come from?

- Classically a shadow would form behind the target:



- In the wave picture the shadow has to be created by destructive interference with the incoming wave e^{ikz}, which is present also behind the target.
- The interfering wave is part of the scattered wave function ψ_s. It is focused in the forward direction and its integrated flux is πR^2.

For a formal treatment we write

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{l_{max}} (2l+1) \underbrace{e^{i\delta_l} \sin(\delta_l)}_{= \frac{1}{2i}\left(e^{2i\delta_l} - 1\right)} P_l(\cos\theta)$$

$$= \underbrace{\frac{1}{2ik} \sum_{l=0}^{l_{max}} (2l+1)\, e^{2i\delta_l} P_l(\cos\theta)}_{f_{refl}} + \underbrace{\frac{i}{2k} \sum_{l=0}^{l_{max}} (2l+1) P_l(\cos\theta)}_{= f_{shadow}}$$

The contribution from the first part to the total cross section is

$$\sigma_{tot}^{refl} = 2\pi \int_{-1}^{1} d(\cos\theta)\, \underbrace{|f_{refl.}(\theta)|^2}$$

$$= \frac{1}{4k^2} \sum_{l,l'} (2l+1)(2l'+1)\, e^{2i(\delta_l - \delta_{l'})} P_l(\cos\theta) P_{l'}(\cos\theta)$$

$$= \frac{\pi}{2k^2} \sum_{l=0}^{l_{max}} 2\cdot(2l+1) \simeq \frac{\pi\, l_{max}^2}{k^2} = \pi R^2 .$$

because of the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{l,l'}$$

The second contribution f_{shadow} is purely imaginary and focused in the forward direction, because all terms are in phase and maximal at $\theta = 0$.

Further considerations show that

$$2\pi \int_{-1}^{1} d(\cos\theta)\, |f_{shadow}(\theta)|^2 \simeq \pi R^2 = \sigma_{tot}^{shadow}$$

and $\sigma_{tot} = \sigma_{tot}^{refl} + \sigma_{tot}^{shadow}$.