4° Photon correlations

In analogy to Section II, 5° we introduce correlation functions for the radiation field.

In this analogy, $E^{(+)}(\vec{r},t)$ corresponds to $\psi(\vec{r})$ and $E^{(-)}(\vec{r},t)$ to $\psi^+(\vec{r})$.

- First order correlation function

$$G^{(1)}(\vec{r},t;\vec{r}',t') = \langle E^{(-)}(\vec{r},t)\, E^{(+)}(\vec{r}',t')\rangle = \mathrm{Tr}(\rho\, E^{(-)}(\vec{r},t)\, E^{(+)}(\vec{r}',t'))$$

For $\vec{r}=\vec{r}'$, $t=t'$ this the field intensity $I(\vec{r},t)$

The first order coherence function is obtained by normalization:

$$g^{(1)}(\vec{r},t;\vec{r}',t') := \frac{G^{(1)}(\vec{r},t;\vec{r}',t')}{\sqrt{G^{(1)}(\vec{r},t;\vec{r},t)\, G^{(1)}(\vec{r}',t',\vec{r}',t')}} = \frac{G^{(1)}(\vec{r},t;\vec{r}',t')}{\sqrt{I(\vec{r},t)\, I(\vec{r}',t')}}$$

- Second order correlation function

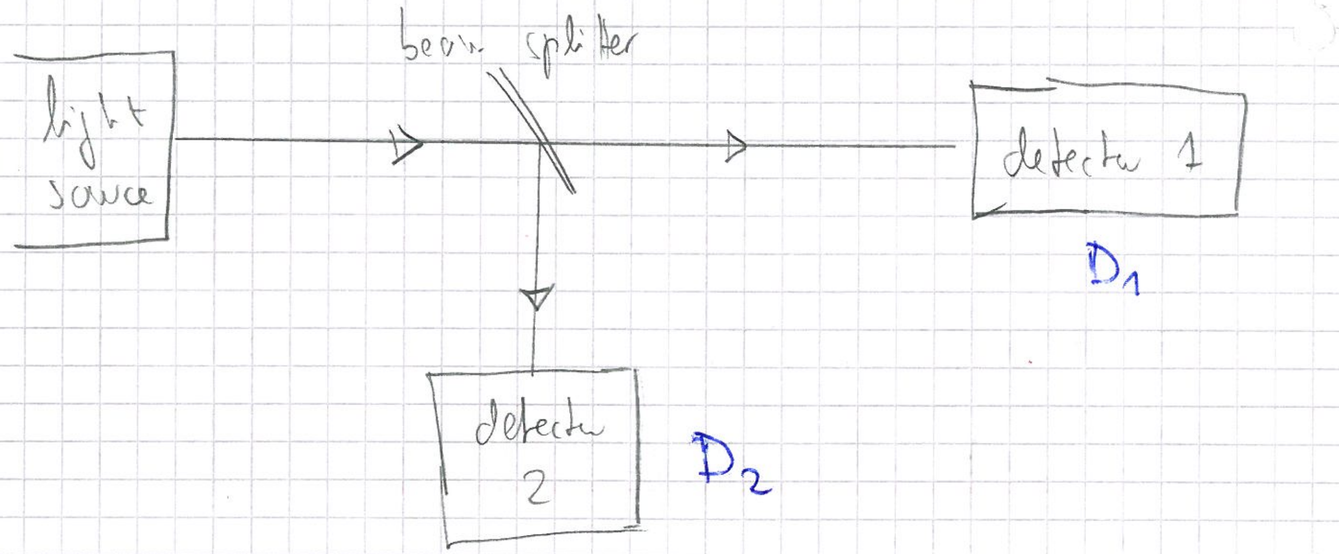
This is analogous to the pair correlation function, and defined by

$$G^{(2)}(\vec{r},t;\vec{r}',t') = \langle E^{(-)}(\vec{r},t)\, E^{(-)}(\vec{r}',t')\, E^{(+)}(\vec{r}',t')\, E^{(+)}(\vec{r},t)\rangle$$

The corresponding second order coherence function

$$g^{(2)}(\vec{r},t;\vec{r}',t') = \frac{G^{(2)}(\vec{r},t;\vec{r}',t')}{I(\vec{r},t)\; I(\vec{r}',t')}$$

measures intensity correlations. A possible realization is the experiment of Hanbury-Brown and Twiss (1957):



The experiment measures intensity correlation as a function of the time delay between the two detectors:

$$g^{(2)} = g^{(2)}(\tau) = \frac{\langle I_1(t)\, I_2(t+\tau)\rangle}{\bar{I}_1 \cdot \bar{I}_2}$$

Examples:

(i) single mode in photon number state $|n_{\vec{k}}\rangle$

$$G^{(1)}(\vec{r},t;\vec{r}',t') = \langle n_{\vec{k}} | E^{(-)}(\vec{r},t) E^{(+)}(\vec{r}',t') | n_{\vec{k}} \rangle$$

$$\left. \begin{aligned} E^{(-)} &= i\sqrt{\frac{2\pi\hbar\omega}{V}}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, a^{+}_{\vec{k}} \\ E^{(+)} &= -i\sqrt{\frac{2\pi\hbar\omega}{V}}\, e^{-i(\vec{k}\cdot\vec{r}'-\omega t')}\, a_{\vec{k}} \end{aligned} \right\}$$

$$\Rightarrow \quad G^{(1)} = \frac{2\pi\hbar\omega}{V}\, n_{\vec{k}}\, \underbrace{e^{-i(\vec{k}\cdot(\vec{r}'-\vec{r}) - \omega(t'-t))}}_{= e^{-i\omega\tau}} \quad \text{with}$$

$$\tau = t - t' - \frac{1}{\omega}\vec{k}\cdot(\vec{r}-\vec{r}')$$

$$\Rightarrow \quad g^{(1)} = e^{-i\omega\tau}, \quad \underline{|g^{(1)}| = 1}$$

$$\underline{G^{(2)}} = \left(\frac{2\pi\hbar\omega}{V}\right)^2 \underbrace{\langle n_{\vec{k}} | a^{+}_{\vec{k}} a^{+}_{\vec{k}} a_{\vec{k}} a_{\vec{k}} | n_{\vec{k}} \rangle}$$

$$= \sqrt{n_{\vec{k}}(n_{\vec{k}}-1)}\, \langle n_{\vec{k}} | a^{+}_{\vec{k}} a^{+}_{\vec{k}} | n_{\vec{k}} - 2 \rangle =$$

$$= \underline{n_{\vec{k}}(n_{\vec{k}}-1)} \quad \text{assuming } n_{\vec{k}} \geq 1$$

$$\Rightarrow \quad \underline{g^{(2)} = \frac{n_{\vec{k}}-1}{n_{\vec{k}}} = 1 - \frac{1}{n_{\vec{k}}} < 1}$$

This is identical to the result for non-interacting bosons with a sharp momentum in Section III. 1° and can be understood from a simple probabilistic picture based on the Hanbury-Brown and Twiss experiment:

- state contains n photons, n_1 of which are detected by D_1, n_2 by D_2
- the probability for the beam splitter to transmit a photon is p
- then the probability for (n_1, n_2) photons to be observed at the two detectors is

$$\underline{P_n(n_1, n_2) = \binom{n}{n_1} p^{n_1} (1-p)^{n_2}} \quad , \; n = n_1 + n_2$$

<u>binomial distribution</u>
(compare Sect. II. 1°)

The first two moments of this distribution are

$$\langle n_1 \rangle = pn \, , \quad \langle n_1^2 \rangle = n^2 p^2 + \underbrace{n p (1-p)}_{\text{variance}}$$

$$\Rightarrow \underline{\langle n_1 n_2 \rangle} = \langle n_1 (n - n_1) \rangle = n \langle n_1 \rangle - \langle n_1^2 \rangle =$$

$$= n^2 p - n^2 p^2 - np - np^2 = \underline{n(n-1)\, p(1-p)}$$

$$\Rightarrow \underline{g^{(2)}} = \frac{\langle n_1 n_2 \rangle}{\langle n_1 \rangle \langle n_2 \rangle} = \frac{n(n-1)}{n^2} = \underline{1 - \frac{1}{n}}$$

This simply expresses the fact that every photon can be detected only once.

(ii) <u>Single mode in a mixed state</u>

Assume $\rho = \sum_n p_n |n\rangle\langle n|$, then

$$G^{(1)} = \mathrm{Tr}\left(\rho E^{(-)} E^{(+)}\right) = \frac{2\pi\hbar\omega}{V} e^{-i\omega\tau} \sum_n n p_n$$

$$= \frac{2\pi\hbar\omega}{V} \bar{n} e^{-i\omega\tau}$$

and $|g^{(1)}| = 1$ as before. Similarly

$$G^{(2)} = \left(\frac{2\pi\hbar\omega}{V}\right)^2 \sum_n p_n \left(n^2 - n\right) =$$

$$= \left(\frac{2\pi\hbar\omega}{V}\right)^2 \left(\langle n^2\rangle - \bar{n}\right)$$

where $\langle n^2\rangle$ is the second moment of p_n.

$$\Rightarrow \quad g^{(2)} = \frac{\langle n^2\rangle - \bar{n}}{(\bar{n})^2} =$$

For a chaotic light field p_n is geometric (Sec. IV. 2°) and therefore

$$\langle n^2\rangle = 2\bar{n}^2 + \bar{n} \quad \Rightarrow \quad g^{(2)} = 2$$

<u>Poisson:</u> $\langle n^2\rangle = \bar{n}^2 + \bar{n} \quad \Rightarrow \quad g^{(2)} = 1$

(iii) Single mode in a coherent state

$$G^{(1)} = \frac{2\pi\hbar\omega}{V} e^{-i\omega\tau} \underbrace{\langle\alpha| a_{\vec{k}}^{+} a_{\vec{k}} |\alpha\rangle}_{=|\alpha|^2}$$

$$\Rightarrow \quad g^{(1)}(\tau) = e^{-i\omega\tau}, \quad |g^{(1)}| = 1$$

$$G^{(2)} = \left(\frac{2\pi\hbar\omega}{V}\right)^2 \underbrace{\langle\alpha| a_{\vec{k}}^{+} a_{\vec{k}}^{+} a_{\vec{k}} a_{\vec{k}} |\alpha\rangle}_{=|\alpha|^4} = |G^{(1)}|^2$$

$$\Rightarrow \quad g^{(2)} = 1$$

(iv) States with multiple modes

For a general photon number state with occupation numbers $\{n_{\vec{k}}\}$

$$G^{(1)}(\vec{r},t;\vec{r}',t') = \frac{2\pi\hbar}{V} \sum_{\vec{k},\vec{k}'} \sqrt{\omega_{\vec{k}}\,\omega_{\vec{k}'}} \times$$

$$\times e^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}\, e^{-i(\vec{k}'\cdot\vec{r}'-\omega_{\vec{k}'}t')} \underbrace{\langle\{n_{\vec{q}}\}| a_{\vec{k}'}^{+} a_{\vec{k}} |\{n_{\vec{q}}\}\rangle}_{=n_{\vec{k}}\,\delta_{\vec{k}\vec{k}'}}$$

$$= \frac{2\pi\hbar}{V} \sum_{\vec{k}} n_{\vec{k}}\, \omega_{\vec{k}}\, e^{-i[\vec{k}\cdot(\vec{r}'-\vec{r}) - \omega_{\vec{k}}(t'-t)]}$$

Assume that all wave vectors are aligned along the x-direction, $\vec{k} = k\,e_x$. Then

$$G^{(1)}(\vec{r},t;\vec{r}',t') = \frac{2\pi\hbar}{V} \sum_k n_k c|k| e^{-i\omega\tau}$$

$$\simeq \frac{2\pi\hbar}{V} \cdot \frac{L}{2\pi} \cdot 2 \int_0^\infty dk\, ck n_k e^{-ick\tau}$$

with $\tau = t - t' - \frac{1}{c}(x - x')$.

Consider in particular a Lorentz spectrum with energy density

$$n_k \hbar\omega = I_0 \frac{\gamma/\pi}{(\omega_0 - \omega)^2 + \gamma^2}$$

where γ is the line width. Then for $\omega_0 \gg \gamma$

$$g^{(1)}(\tau) = e^{-i\omega_0\tau} e^{-\gamma|\tau|}$$

$\Rightarrow |g^{(1)}(\tau)|$ decays exponentially on the time scale $1/\gamma$.

This result holds also for a chaotic light field with mean photon number $\bar{n}_{\vec{k}}$. In that case the second order coherence function is simply

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

Summary

- $|g^{(1)}|$ and $g^{(2)}$ are constant for single mode fields.

- $|g^{(1)}| = 1$ always, whereas $g^{(2)}$ reflects the statistics of photon counting events. There are three distinct cases:

(i) $g^{(2)} < 1$: The detection of a pair of photons is less likely than if the photons were independent (antibunching)

(ii) $g^{(2)} > 1$: Photons bunch; e.g. in a chaotic light field $g^{(2)} = 2$

(iii) $g^{(2)} = 1$: Photons are effectively independent. This applies to coherent states which are approximately realized by lasers.

- For multimode fields $g^{(1)}$ and $g^{(2)}$ decay on a time scale determined by the spectral line width. In this case the photon counting statistics is determined by $g^{(2)}(0)$.

- Antibunching is a purely quantum mechanical effect, because in any classical theory

$$g^{(2)}(0) = \frac{\langle I(\vec{r},t)^2 \rangle}{\langle I(\vec{r},t) \rangle^2} \geq 1 .$$

V. Relativistic quantum mechanics

Goal: Develop a single particle description that is consistent with special relativity.

- Because of the inevitable appearance of negative energy states associated with antiparticles, it will turn out that the single particle description is actually inconsistent.
- Nevertheless in a low energy approximation we will be able to understand the origin of spin and obtain an improved description of atomic spectra.

Relativistic notation

(i) Space-time points are described by four-vectors

$$x^\mu = (ct, \underbrace{x_1, x_2, x_3}_{x,y,z}) \quad \text{contravariant}$$

$$x_\mu = (ct, -x_1, -x_2, -x_3) \quad \text{covariant}$$

which are related by

$$x_\mu = \sum_{\nu=1}^{4} g_{\mu\nu}\, x^\nu =: g_{\mu\nu}\, x^\nu \qquad \text{summation convention}$$

with the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

The Minkowski distance

$$x_\mu x^\mu = c^2 t^2 - x_1^2 - x_2^2 - x_3^2$$

is invariant under Lorentz transformations.

(ii) Energy and momentum are components of the four-vector

$$p^\mu = (p_0, p_1, p_2, p_3) = (E/c, p_x, p_y, p_z)$$

The scalar invariant associated with this vector is associated with the rest energy:

$$p_\mu p^\mu = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 =: m^2 c^2$$

$$\Rightarrow \quad E = \sqrt{c^2 |\vec{p}|^2 + m^2 c^4}$$

is the energy-momentum relation of a free relativistic particle (cf. Chapter III).

(iii) The four-gradient

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

is a covariant vector

1° The Klein-Gordon equation

a) Motivation

Reminder: In non-relativistic quantum mechanics, the Schrödinger equation is obtained through the following steps:

energy-momentum relation $\quad E = \dfrac{|\vec{p}|^2}{2m}$

$E = \hbar\omega \quad \downarrow \quad \vec{p} = \hbar\vec{k}$

dispersion relation $\quad \omega = \dfrac{\hbar |\vec{k}|^2}{2m}$

$E = i\hbar\dfrac{\partial}{\partial t} \quad \downarrow \quad \vec{p} = -i\hbar\nabla$

wave equation $\quad i\hbar\dfrac{\partial\psi}{\partial t} = -\dfrac{\hbar^2}{2m}\nabla^2\psi$

Applying the same approach in the relativistic case yields

$$i\hbar\frac{\partial\psi}{\partial t} = \sqrt{-\hbar^2 c^2\nabla^2 + m^2c^4}\,\psi =$$

$$= \left(mc^2 - \frac{\hbar^2}{2m}\nabla^2 - \frac{1}{8}\frac{\hbar^4}{(mc^2)^3}(\nabla^2)^2 - \frac{1}{16}\frac{\hbar^6}{(mc^2)^5}(\nabla^2)^3 + \ldots\right)\psi$$

which is not Lorentz invariant and moreover non-local in space.

We therefore start instead from the squared energy-momentum relation which is polynomial:

$$E^2 = c^2 |\vec{p}|^2 + m^2 c^4$$

$$\Downarrow$$

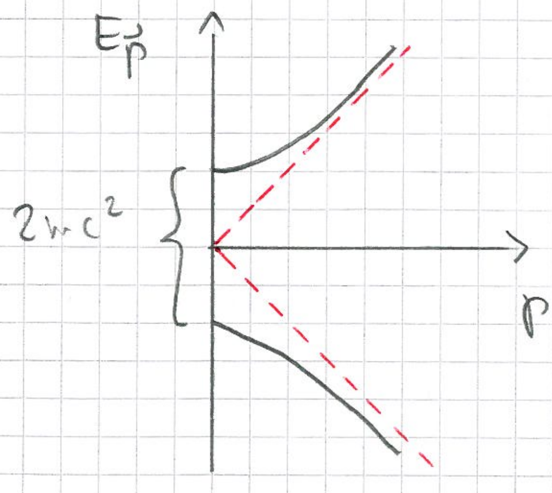
$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \psi$$

$$\Rightarrow \left(\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}_{= \Box \, = \, \partial_\mu \partial^\mu \text{ d'Alembert operator}} + \left(\frac{mc}{\hbar}\right)^2\right) \psi = 0 \qquad \text{KG}$$

Klein-Gordon equation

This equation is manifestly Lorentz-invariant. The free particle solutions are

$$\left.\begin{array}{l} \psi_{\vec{p}}(\vec{r}, t) = A\, e^{-\frac{i}{\hbar}\left(E_{\vec{p}}\, t - \vec{p}\cdot\vec{r}\right)} \\ \text{with } \quad E_{\vec{p}} = \pm\sqrt{c^2 |\vec{p}|^2 + m^2 c^4} \end{array}\right\}$$



How to interpret the solutions with negative energy?

Note: $\left(\frac{mc}{\hbar}\right)^2 = \frac{1}{\lambda_c^2}$

with the Compton length λ_c

b) Continuity equation

Reminder: The Schrödinger equation conserves probability, which implies that the continuity equation holds:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 , \qquad \rho = |\psi|^2 ,$$

$$\vec{j} = -\frac{i\hbar}{2m}\left(\psi^* \nabla\psi - \psi \nabla\psi^*\right)$$

To look for a similar property in the Klein-Gordon theory we consider the four-vector

$$j^\mu = \frac{i\hbar}{2m}\left[\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*\right]$$

We know from charge conservation in electromagnetism that the Lorentz-invariant form of the continuity equation reads

$$\underbrace{\partial_\mu j^\mu}_{\text{four-divergence}} = 0 \qquad \text{with} \qquad j^\mu = (c\rho, \vec{j})$$

Indeed here we have

$$\partial_\mu j^\mu = \frac{i\hbar}{2m}\Big[(\partial_\mu\psi^*)(\partial^\mu\psi) + \psi^* \partial_\mu\partial^\mu\psi - (\partial_\mu\psi)(\partial^\mu\psi^*) - \psi\, \partial_\mu\partial^\mu\psi^*\Big]$$

Using the Klein-Gordon equation

$$\partial_\mu \partial^\mu \psi = -\left(\frac{mc}{\hbar}\right)^2 \psi, \quad \partial_\mu \partial^\mu \psi^* = -\left(\frac{mc}{\hbar}\right)^2 \psi^*$$

and the fact that $(\partial_\mu \psi^*)(\partial^\mu \psi) = (\partial_\mu \psi)(\partial^\mu \psi^*)$ we see that

$$\partial_\mu j^\mu = 0 \quad \Rightarrow \quad \rho = \frac{1}{c} j^0 =$$

$$= \frac{i\hbar}{2mc^2}\left[\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right]$$

is conserved. However, ρ is not generally positive and hence cannot be interpreted as a probability density.

This problem is mathematically related to the fact that the Klein-Gordon-eq. is second order in time. We therefore rewrite the equation as two coupled first order equations for the functions

$$\phi(\vec{r},t) = \frac{1}{2}\left[\psi + \frac{i\hbar}{mc^2}\partial_t \psi\right]$$

$$\chi(\vec{r},t) = \frac{1}{2}\left[\psi - \frac{i\hbar}{mc^2}\partial_t \psi\right]$$

such that

$$\left.\begin{aligned} \phi + \chi &= \psi, \\ \phi - \chi &= \frac{i\hbar}{mc^2}\partial_t \psi \end{aligned}\right\}$$

KG

$$\Rightarrow \quad \partial_t \phi = \frac{1}{2}\left[\partial_t \psi + \frac{i\hbar}{mc^2}\partial_t^2 \psi\right] =$$

$$= \frac{1}{2}\left[-\frac{imc^2}{\hbar}(\phi - \chi) + \frac{i\hbar}{m}\left(\nabla^2 - \frac{m^2c^4}{\hbar^2}\right)(\phi+\chi)\right]$$

$$= \frac{i\hbar}{2m}\nabla^2(\phi+\chi) - \frac{imc^2}{\hbar}\phi$$

$$\Rightarrow \quad i\hbar\, \partial_t \phi = -\frac{\hbar^2}{2m}\nabla^2(\phi+\chi) + mc^2\phi$$

For $\chi = 0$ this is the Schrödinger eq. for a particle with rest energy mc^2. Similarly χ satisfies

$$i\hbar\, \partial_t \chi = \frac{\hbar^2}{2m}\nabla^2(\phi+\chi) - mc^2\chi$$

For $\phi = 0$ this is the Schrödinger equation for a particle moving "backward in time" $(t \to -t)$.

In terms of ϕ and χ, the density ρ is

$$\rho = \frac{i\hbar}{2mc^2}\left[\psi^* \frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right] = |\phi|^2 - |\chi|^2$$

Interpretation: ρ is the "charge density", where ϕ and χ describe particles and antiparticles of opposite charge.

Exercise: Free particle solutions for ϕ and χ.