Institute of Theoretical Physics — A. Rosch
University of Cologne — L. Heinen, V. Lohani, H. F. Legg, J. Park
Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 1

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. The Klein-Gordon equation

Consider the field theory of a complex field obeying the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi).$$

ϕ and ϕ^* can be treated as independent fields.

a) Derive the Klein-Gordon equation using the Euler-Lagrange equation of motion for the fields.

b) Show the continuity equation of $\partial_\mu j^\mu = 0$ for the current $j^\mu = i\left[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)\right]$.

c) Find the conjugate momenta $\Pi = \partial\mathcal{L}/\partial(\partial_t \phi)$ and $\Pi^* = \partial\mathcal{L}/\partial(\partial_t \phi^*)$ to $\phi(x)$ and $\phi^*(x)$, respectively. Construct the Hamiltonian density $\mathcal{H} \equiv \Pi(\partial_t \phi) + \Pi^*(\partial_t \phi^*) - \mathcal{L}$ in terms of ϕ, Π and ϕ^*, Π^* and their derivatives.

2. The Dirac equation

Use the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0,$$

to prove the continuity equation $\partial_\mu j^\mu = 0$ for the current $j^\mu = \bar{\psi}(x)\gamma^\mu \psi(x)$.

3. Generator of the Lorentz group (*)

a) The Lorentz group is the (Lie-)group of rotations and boosts Λ in space-time. Generally, such a Lie group is *generated* by the exponential of its corresponding generators J, i.e.,

$$\Lambda = e^{i\phi J} = 1 + i\phi J + O(\phi^2).$$

J^{0i} generate boosts in direction x_i and J^{ij} generates rotations around the axis perpendicular to x_i and x_j $(i \neq j)$. For the boost in the x direction, Λ can be written as

$$\Lambda = \begin{pmatrix} \cosh(\phi) & \sinh(\phi) & 0 & 0 \\ \sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

ϕ is called rapidity. What is the generator J^{01} of this transformation? How does ϕ depend on v/c?

b) The generator $J^{\mu\nu}$ fulfills the Lorentz-algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

The generators which describe the transformation properties ($\psi \to e^{i\phi S}\psi$) of 4-component spinors are

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Show that it fullfills the Lorentz-algebra. You may use identities

$$[A, BC] = [A, B]C + B[A, C], \qquad [A, BC] = \{A, B\}C - B\{A, C\}.$$

and a relation of γ matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

c) The spinor for a particle at rest is given by

$$\psi(x) = e^{-imt}\begin{pmatrix}\xi \\ \xi\end{pmatrix}, \tag{2}$$

where ξ is an arbitrary two-component spinor. What is the spinor for a particle moving in x direction with rapidity ϕ?