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Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 10

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. The Josephson tunnel junction (*)

Consider two superconducting quantum dots separated by a tunneling barrier. Each dot is described by the action

$$S^i[\bar{\Psi}, \Psi, \phi] = \int_0^\beta d\tau \Big[\sum_\alpha \bar{\Psi}^i_\alpha \left[\partial_\tau + \xi_{i\alpha}\sigma_3 + \Delta\left(\sigma_1 \cos\phi_i + \sigma_2 \sin\phi_i\right)\right] \Psi^i_\alpha \Big] \tag{1}$$

where $\Psi^i_\alpha = (\psi^i_{\alpha\uparrow}, \bar{\psi}^i_{\alpha\downarrow})^T$ are Nambu spinors, the index i labels the dot, α labels single-particle states of the system, Δ is the superconducting gap that is assumed to be constant, σ_i Pauli matrices in particle-hole space, and $\phi_i(\tau)$ are the phase of the superconducting order parameter on dot i.
We assume that the interaction between dots is "capacitative"

$$S_{\text{int}} = \frac{E_C}{4} \int d\tau (N_1 - N_2)^2, \tag{2}$$

where N_1 and N_2 are the numbers of electrons in dots 1 and 2 (expressed with the Nambu spinors as $N_i = \sum_\alpha \bar{\Psi}^i_\alpha \sigma_3 \Psi^i_\alpha$). Finally, the tunneling term between the two dots can be written as

$$S_T = \sum_{\alpha\beta} \int d\tau \bar{\Psi}^1_\alpha t_{\alpha\beta} \Psi^2_\beta + \text{H.c.} \tag{3}$$

a) Use a Hubbard-Stratonovich transformation (H-S field V) to get rid of the interaction term. Then, use a gauge transformation

$$\Psi \to \Psi' = \begin{pmatrix} e^{i\phi_1\sigma_3/2} & 0 \\ 0 & e^{i\phi_2\sigma_3/2} \end{pmatrix} \Psi \tag{4}$$

to remove the phase dependence from the superconducting order parameter. Derive the effective action

$$S_{\text{eff}}[V, \phi_1, \phi_2] = S_c[V] - \text{Tr}\log\left[-\hat{G}^{-1}\right]$$

with

$$\hat{G}^{-1} = \begin{pmatrix} \partial_\tau + (\xi_{1\alpha} + i(\dot{\phi_1} + V)/2)\sigma_3 + \Delta\sigma_1 & e^{-i(\phi_1-\phi_2)\sigma_3/2} t \\ t^\dagger e^{+i(\phi_1-\phi_2)\sigma_3/2} & \partial_\tau + (\xi_{2\alpha} + i(\dot{\phi_2} - V)/2)\sigma_3 + \Delta\sigma_1 \end{pmatrix}$$

and $S_c = \int d\tau V^2/(4E_C)$.

b) We now want to find a saddle point for ϕ_1 and ϕ_2 with arbitrary V which we parameterize by $V = \dot{\phi}$. For vanishing hopping $t_{\alpha\beta}$, set

$$\begin{aligned}\phi_1 &= -\phi + \delta\phi_1, \\ \phi_2 &= \phi + \delta\phi_2,\end{aligned}$$

and show that terms linear in $\delta\phi_1$ and $\delta\phi_2$ have the form of $\int_0^\beta d\tau\dot{\phi} \times \text{const} = 0$ and vanish.

c) Set approximately $\delta\phi_i = 0$. (To get a more accurate result one can alternatively integrate out $\delta\phi_i$ perturbatively.) The mean-field solution implies the so-called *Josephson condition*:

$$\dot{\phi}_2 - \dot{\phi}_1 = 2V = 2\dot{\phi}.$$

Expand the action to second order in the tunneling and show that $S = S_c + S_{\text{tun}}[\phi] + S_0$, where

$$S_{\text{tun}}[\phi] = |t|^2 \sum_{\alpha\alpha'} \int d\tau_1 d\tau_2 \text{Tr}\left(G_{1\alpha}(\tau_1-\tau_2)e^{i\sigma_3\phi(\tau_2)}G_{2\alpha'}(\tau_2-\tau_1)e^{-i\sigma_3\phi(\tau_1)} - G_{1\alpha}(\tau_1-\tau_2)G_{2\alpha'}(\tau_2-\tau_1)\right). \tag{5}$$

Here $G_{i\alpha}(\tau)$ is the Fourier transform of $G_{i\alpha\omega_n} = (i\omega_n - \xi_{i\alpha}\sigma_3 - \Delta\sigma_1)^{-1}$. S_0 is the action independent of ϕ. We assume that the hopping parameters $t_{\alpha\beta}$ do not depend on the single-particle states.

d) Let us calculate $S_{\text{tun}}[\phi]$. First, notice that the Green's function has both a diagonal ($G_{i\alpha,d}$) and an off-diagonal ($G_{i\alpha,o}$) component. It turns out that up to small corrections, the diagonal terms vanish in (5), so you may safely concentrate on the off-diagonal components.
Noting that $G_{i,o}(\tau)$ has the form of $G_{i,o}(\tau) = \sigma_1\tilde{G}_{i,o}(\tau)$ ($\tilde{G}_{i,o}(\tau)$ is the identity in the particle-hole space) and using the assumption of a low-frequency limit of ϕ compared with the gap Δ satisfying $\sum_{\alpha\alpha'}\tilde{G}_{1\alpha,o}(\tau)\tilde{G}_{2\alpha',o}(-\tau) = \sum_{\alpha\alpha'}\delta(\tau)\int d\tau_1\tilde{G}_{1\alpha,o}(\tau_1)\tilde{G}_{2\alpha',o}(-\tau_1) \equiv A\delta(\tau)$, derive the celebrated action for a Josephson junction of

$$\frac{1}{4E_C}\int d\tau\dot{\phi}^2 + \gamma\int d\tau\cos(2\phi(\tau)). \tag{6}$$

Here $\gamma = 2|t|^2 A$.

2. The Ginzburg-Landau action

Let's consider a more generalized version of the Ginzburg-Landau action of

$$S = \int d\tau d^3\boldsymbol{r}\left(\frac{r}{2}|\Delta|^2 + \frac{1}{2m}\left(|\boldsymbol{\nabla}\Delta|^2 + c^2|\partial_\tau\Delta|^2\right) + \frac{u}{4}|\Delta|^4 + \alpha\Delta^*\partial_\tau\Delta\right). \tag{7}$$

Note that the last term is added compared with the lecture. In the decomposition of $\Delta = (\Delta_0 + \delta)e^{i\phi}$ with a real Higgs field δ and a goldstone mode ϕ, expand the action to the second order in δ and ϕ. Here the time and position-independent Δ_0 is set to the value to minimize the Mexican-hat potential. What is the new term coming from the last term? Prove that the goldstone mode ϕ is still massless.