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Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 11

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. Scaling laws

We derive scaling laws by simple scaling arguments. Let us consider the following scaling ansatz for the free energy per volume f as a function of temperature T and magnetic field B,

$$f(T,B) = b^{\lambda_f} \tilde{f}(tb^{\lambda_t}, Bb^{\lambda_b}), \tag{1}$$

where t is defined as $t \equiv (T - T_c)/T_c$, λ_t, λ_b, and λ_s are some exponents, and b is the parameter to describe the scaling of a microscopic length parameter $a \to ba$. Critical exponents $\alpha, \beta, \gamma, \delta$ of the specific heat C, the magnetization M approaching the transition temperature from below, the susceptibility χ, and the field dependence of the magnetization are, respectively, defined as

$$\begin{aligned} C \equiv -T^2 \frac{\partial^2 f}{\partial T^2}\bigg|_{B=0} \sim |t|^{-\alpha}, \;\; M \equiv -\frac{\partial f}{\partial B}\bigg|_{B=0} \simeq (-t)^{\beta}, \\ \chi \equiv -\frac{\partial^2 f}{\partial B^2}\bigg|_{B=0} \simeq |t|^{-\gamma}, \;\; M(t=0) \equiv -\frac{\partial f}{\partial B}\bigg|_{t=0} \simeq |B|^{1/\delta}. \end{aligned} \tag{2}$$

Express the exponents λ_t, λ_b, and λ_s in terms of $\alpha, \beta, \gamma, \delta$ using the scailing ansatz. Derive the scaling laws, the Rushbrooke scaling

$$\alpha + 2\beta + \gamma = 2, \tag{3}$$

and the Widom scaling

$$\beta(\delta - 1) = \gamma. \tag{4}$$

2. Scaling close to a quantum critical point (*)

Close to a quantum phase transition, certain quantities can be obtained by simple scaling arguments. The goal of the present exercise is to calculate the "Grüneisen-ratio" $\Gamma = \alpha/c_p$, where $\alpha = \frac{1}{V}\frac{\partial V}{\partial T}\mid_{p,N} = -\frac{1}{V}\frac{\partial S}{\partial p}\mid_{T,N}$ is the thermal expansion and $c_p = \frac{T}{N}\frac{\partial S}{\partial T}\mid_p$ is the molar specific heat. The Grüneisen-ratio is an important criterion to experimentally verify the presence or absence of a quantum critical point.

a) Our starting point is the following scaling ansatz for the free energy per volume f:

$$f = -b^{-(d+z)} \tilde{f}\left(\frac{T}{T_0} b^z, r b^{1/\nu}\right)$$

where d is the dimension, T the temperature, T_0 a non-universal reference temperature, $r = (p - p_c)/p_c$ the control parameter of the phase transition (at the critical pressure p_c), ν and z are the critical exponents of the correlation length ξ and correlation time ξ_τ, i.e.

$$\xi \propto |r|^{-\nu}, \quad \xi_\tau \propto \xi^z.$$

The parameter b can be set to an arbitrary value. $\tilde{f}$ is a unknown universal scaling function.
Show that by appropriate choice of b, the free energy can brought into the following two alternative forms:

$$\begin{aligned} f &= -r^{\nu(d+z)} \tilde{f}_1\left(\frac{T}{T_0 r^{\nu z}}\right) \\ f &= -\left(\frac{T}{T_0}\right)^{(d+z)/z} \tilde{f}_2\left(\frac{r}{(T/T_0)^{1/(\nu z)}}\right) \end{aligned}$$

where $\tilde{f}_1, \tilde{f}_2$ are two universal scaling functions.

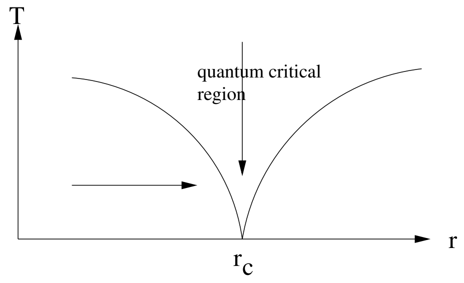

Figure 1: Schematic phase diagram of a quantum phase transition. The two arrows show two different ways to experimentally enter the quantum critical regime at finite temperature.

b) First, we want to approach the quantum critical point (QCP) $(T = 0, p = p_c)$ as $r = 0$ and $T > 0$ (see the arrow pointing down in Fig. 1). Calculate α and c_p from one of the above representations of the free energy.

c) Second, we approach the QCP as $T \to 0, r \neq 0$ (e.g. see the arrow pointing right in Fig. 1). Here, we assume that $\tilde{f}_1(x \to 0) = \tilde{f}_1(0) + c x^{y_0+1} + ...$, where y_0 is a material-dependent positive number. Calculate α and c_p.

d) Calculate the Grüneisen-ratio in both directions, obtaining the following result:

$$\Gamma = \begin{cases} -G_T T^{-1/(\nu z)} & T \neq 0, r = 0 \\ -G_r \frac{N}{V(p-p_c)} & T \to 0, r \neq 0 \end{cases}$$

What are G_T, G_r? In what sense is the second result more universal than the first one?

A foonote: If you are further interested in some background information, see *arXiv:cond-mat/0212335*.