Institute of Theoretical Physics A. Rosch
University of Cologne L. Heinen, V. Lohani, H. F. Legg, J. Park
Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 12

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. N-component ϕ^4-model (*)

Consider the ϕ^4-theory in dimension $D = 4 - \epsilon$, defined by the Euclidean action:

$$S_\phi = \int d^D x \left[\frac{1}{2} \sum_\alpha (\partial_x \phi_\alpha)^2 + \frac{1}{2} r \sum_\alpha \phi_\alpha^2 + \frac{\lambda}{4!} \sum_{\alpha\beta} \phi_\alpha^2 \phi_\beta^2 \right], \tag{1}$$

with field ϕ having N components, $\phi = (\phi_1, ..., \phi_N)$. In the lecture, it was shown that for $N = 1$, the application of the RG procedure and integrating out the fast components using perturbation theory to leading order give

$$\frac{dr}{d\ln b} = 2r + \frac{\lambda}{16\pi^2 (r+1)}, \tag{2}$$

$$\frac{d\lambda}{d\ln b} = \epsilon\lambda - \frac{3\lambda^2}{16\pi^2 (r+1)^2}. \tag{3}$$

a) Find the generalized expression of these equations for the N-component model. In order to do so, split every component in a slow and a fast part. The calculations for an arbitrary N are analogous to the $N = 1$ case. The difference is in the N dependent prefactors of diagrams that contribute to the final result within our perturbative approach; One has to keep track on the summations over internal indices α, β. Find the diagrams that yield corrections to the parameters r, λ and derive the generalized expressions

$$\frac{dr}{d\ln b} = 2r + \lambda \frac{N+2}{48\pi^2 (r+1)}, \tag{4}$$

$$\frac{d\lambda}{d\ln b} = \epsilon\lambda - \lambda^2 \frac{N+8}{48\pi^2 (r+1)^2}, \tag{5}$$

Note that in order to get the correct result no explicit calculation needs to be done. The generalized prefactors are purely determined by correctly counting the number of diagrams.

2. Coupled scalars

Consider the action

$$S = \int \mathrm{d}^D \boldsymbol{x} \left[\frac{t}{2} m^2 + \frac{K}{2} (\nabla m)^2 - h\, m + \frac{L}{2} \left(\nabla^2 \phi\right)^2 + v \nabla m \cdot \nabla \phi \right], \tag{6}$$

in terms of two one-component fields m and ϕ.

a) By rescaling distances, $\boldsymbol{x}' = \boldsymbol{x}/b$ and fields, $m'(\boldsymbol{x}') = m(\boldsymbol{x}) b^{\lambda_m}$ and $\phi'(\boldsymbol{x}') = \phi(\boldsymbol{x}) b^{\lambda_\phi}$, check how parameters K, t, h, L, and v are renormalized.

b) Choose a rescaling scheme where the renormalized parameters K' and L' satisfy $K' = K$ and $L' = L$. What are the renormalization group (RG) equations for t, h, and v? Which operators are relevant? Repeat the analysis for a rescaling scheme where $t' = t$ and $L' = L$. What are in this case the RG equations for K, h, and v?

c) Consider the scaling form of the free energy

$$f(t, h, v) = b^{\lambda_f} \tilde{f}(t b^{\lambda_t}, h b^{\lambda_h}, v b^{\lambda_v}), \tag{7}$$

from the rescaling scheme of $K' = K$ and $L' = L$. Here $\lambda_f = -D$ through the hyperscaling hypothesis. Using the scaling form, derive the form $f(t, h, v) = t^{2-\alpha} \tilde{f}(h/t^{\Delta}, v/t^{\omega})$ for t, h, v close to zero. Find α, Δ and ω in terms of the dimension D using the RG equations obtained in b).