Institute of Theoretical Physics A. Rosch
University of Cologne L. Heinen, V. Lohani, H. F. Legg, J. Park
Exercise sheet online at http://www.thp.uni-koeln.de/~jinhong/qft2

Quantum Field Theory II
Problem Set 13

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. The relevance of various couplings for massless particles (*)

The goal of this exercise is to check whether some couplings are relevant, irrelevant, or marginal on the tree level in different spatial dimensions $D = 1, 2, 3$. Let us consider the fixed point Euclidean action for two-component (let say spin up and down) massless particles

$$S_0 = \int d\tau \int d^D\boldsymbol{r} \Psi^\dagger \left(\partial_\tau - iv\boldsymbol{\sigma}\cdot\nabla\right)\Psi. \tag{1}$$

Here v is the velocity of the particles. We add two types of interactions, the local interaction and the long-ranged Coulomb interaction, as

$$\begin{aligned} S_{\text{int}} =& g\int d\tau \int d^D\boldsymbol{r}\Psi^\dagger_\uparrow(\boldsymbol{r},\tau)\Psi_\uparrow(\boldsymbol{r},\tau)\Psi^\dagger_\downarrow(\boldsymbol{r},\tau)\Psi_\downarrow(\boldsymbol{r},\tau) \\ &+ \alpha\sum_{\alpha,\beta=\uparrow,\downarrow}\int d\tau\int d^D\boldsymbol{r}\int d^D\boldsymbol{r'}\Psi^\dagger_\alpha(\boldsymbol{r},\tau)\Psi^\dagger_\beta(\boldsymbol{r'},\tau)\frac{1}{|r-r'|}\Psi_\beta(\boldsymbol{r'},\tau)\Psi_\alpha(\boldsymbol{r},\tau) \end{aligned} \tag{2}$$

a) Determine whether the interactions are relevant, irrelevant, or marginal on the tree level in dimensions $D = 1, 2, 3$, respectively. If interactions are irrelevant this implies that weak interaction will not modify low-energy properties. Only large interactions can, e.g., introduce a phase transition.

b) We now consider static random disorder, described by the following action

$$S_{\text{dis}} = \sum_{\alpha=\uparrow,\downarrow} \int d\tau \int d^D\boldsymbol{r} V(r) \Psi_\alpha^\dagger(\boldsymbol{r},\tau)\Psi_\alpha(\boldsymbol{r},\tau). \tag{3}$$

$V(\boldsymbol{r})$ is the real disorder parameter that does not depend on τ and satisfies a Gaussian correlation as

$$\langle V(\boldsymbol{r})V(\boldsymbol{r}')\rangle_{\text{dis}} = W\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{4}$$

Here W is the variance of the disorder and $\langle\rangle_{\text{dis}}$ stands for the averaging over disorder realization. Then, the distribution function of the disorder $P[V]$ reads

$$P[V] = Ce^{-\int d^D\boldsymbol{r} V(\boldsymbol{r})^2/(2W)}. \tag{5}$$

Here C is the normalization factor that satisfies $\int DVP[V] = 1$. We are interested in the average value of some observable $\hat{O}$. The average should be taken over the average over both the field configuration and the disorder realization. The average value is written as

$$\left\langle \hat{O} \right\rangle = \int DVP[V] \left(\frac{\int D\Psi D\Psi^\dagger e^{-S_0[\Psi,\Psi^\dagger]-S_{\text{dis}}[\Psi,\Psi^\dagger,V]}O}{\int D\Psi D\Psi^\dagger e^{-S_0[\Psi,\Psi^\dagger]-S_{\text{dis}}[\Psi,\Psi^\dagger,V]}} \right). \tag{6}$$

This is very difficult to evaluate as $V(\boldsymbol{r})$ shows up in the denominator. To avoid this problem, one uses the so-called replica trick by writing the denominator as

$$\frac{1}{\int D\Psi D\Psi^\dagger e^{-S_0[\Psi,\Psi^\dagger]-S_{\text{dis}}[\Psi,\Psi^\dagger,V]}} = \left(\int D\Psi D\Psi^\dagger e^{-S_0[\Psi,\Psi^\dagger]-S_{\text{dis}}[\Psi,\Psi^\dagger,V]} \right)^{N-1}. \tag{7}$$

The trick is that for positive integer N the problem does not have any denominator anymore. One calculates the observable for all *positive* integer N. At the very end of the calculation (we will not have to do this for this problem) N is sent to zero by analytical continuation. The Nth power of the partition sum is simply obtained by taking N copies of the fields denoted by Ψ_i, $i = 1, \ldots, N$. Then the average value of $\hat{O}$ is given for $N \geq 1$ by

$$\left\langle \hat{O} \right\rangle = \int DVP[V]D\Psi D\Psi^\dagger O e^{-\sum_{i=1}^N \left(S_0[\Psi_i,\Psi_i^\dagger]+S_{\text{dis}}[\Psi_i,\Psi_i^\dagger,V]\right)}. \tag{8}$$

Integrating over the disorder, show that the effective action is written as

$$S_{\text{eff}} = \sum_{i=1}^N S_0[\Psi_i,\Psi_i^\dagger] - \frac{W}{2}\sum_{i=1}^N\sum_{j=1}^N\sum_{\alpha,\beta=\uparrow,\downarrow} \int d^D\boldsymbol{r} \int d\tau \int d\tau' \Psi_{i,\alpha}^\dagger(\boldsymbol{r},\tau)\Psi_{i,\alpha}(\boldsymbol{r},\tau)\Psi_{j,\beta}^\dagger(\boldsymbol{r},\tau')\Psi_{j,\beta}(\boldsymbol{r},\tau') \tag{9}$$

Using this effective action, determine whether the disorder is relevant, irrelevant, or marginal on the tree level in dimensions $D = 1, 2, 3$, respectively.

2. Renormalization group flows

Sketch the renormalization group (RG) flow and determine possible fixed points for the below RG equations:

a)

$$\frac{\partial g}{\partial \ln b} = g + g^2. \tag{10}$$

b)

$$\frac{\partial g}{\partial \ln b} = -g + g^2. \tag{11}$$

c) For the ϕ^4 model in small r and λ limit,

$$\frac{\partial r}{\partial \ln b} = 2r + \frac{\lambda}{16\pi^2}, \tag{12}$$

$$\frac{\partial \lambda}{\partial \ln b} = \epsilon\lambda - \frac{3\lambda^2}{16\pi^2}. \tag{13}$$

ϵ is assumed to be a positive value.

d)

$$\frac{\partial g_1}{\partial \ln b} = g_2, \tag{14}$$

$$\frac{\partial g_2}{\partial \ln b} = -g_1. \tag{15}$$

What changes when you add $g_1^2 + g_2^2$ in the right hand side of both the coupled equations? What changes when you add, instead, $-(g_1^2 + g_2^2)$ in the right hand side of both the coupled equations?

e) For the Kosterlitz-Thouless transition,

$$\frac{\partial g_1}{\partial \ln b} = 2g_2^2, \tag{16}$$

$$\frac{\partial g_2}{\partial \ln b} = 2g_1 g_2. \tag{17}$$