Institute of Theoretical Physics A. Rosch
University of Cologne L. Heinen, V. Lohani, H. F. Legg, J. Park
Exercise sheet online at http://www.thp.uni-koeln.de/~jinhong/qft2

Quantum Field Theory II
Problem Set 2

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

---

1. Left- and right-handed spinors

The free Dirac Lagrangian density reads:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m(\bar{\psi}\psi). \tag{1}$$

a) Show that $P_{L/R} = \frac{1}{2}(\mathbf{1} \pm \gamma^5)$ is a projector (i.e. $P^2 = P$) and $P\gamma^0 P = 0$. γ^5 is defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

b) The right- and left-handed spinors are defined as

$$\psi_{L/R} = \frac{1}{2}(\mathbf{1} \pm \gamma^5)\psi$$

and hence $\psi = \psi_L + \psi_R$. We will need this construction when discussing week interaction and the Higgs mechanism. For massless particles, the Lagrangian density decouples right- and left-handed particles, i.e.

$$\mathcal{L}_{m=0}[\psi_L + \psi_R] = \mathcal{L}_{m=0}[\psi_L] + \mathcal{L}_{m=0}[\psi_R]$$

This implies that the handedness of massless particles is an additional conserved quantity. Show that for the massless case, spin and momentum are (anti-)parallel to each other for a right (left) handed spinor.

c) However, show that the mass term in the Lagrangian density couples right- and left-handed spinors .

2. "Rest energy" of a classical point particle

We shall calculate the "rest energy" of a pointlike unit charge in classical electrodynamics, which is the energy of the electrostatic field created by the charge,

$$E_r = \frac{1}{2}\int d^3\mathbf{r}|\mathbf{E}(\mathbf{r})|^2. \tag{2}$$

a) Calculate the "rest energy" in real space. Use the Coulomb potential $\phi(\mathbf{r}) = e/(4\pi r)$ and the fine structure constant $\alpha = e^2/(4\pi)$. Since the integral becomes divergent at small r (i.e. at large momenta, called "UV" divergence), replace the lower boundary by r_0 where $r_0 \to 0$. Show that the rest energy diverges, $E_r = \alpha r_0^{-1}/2$.

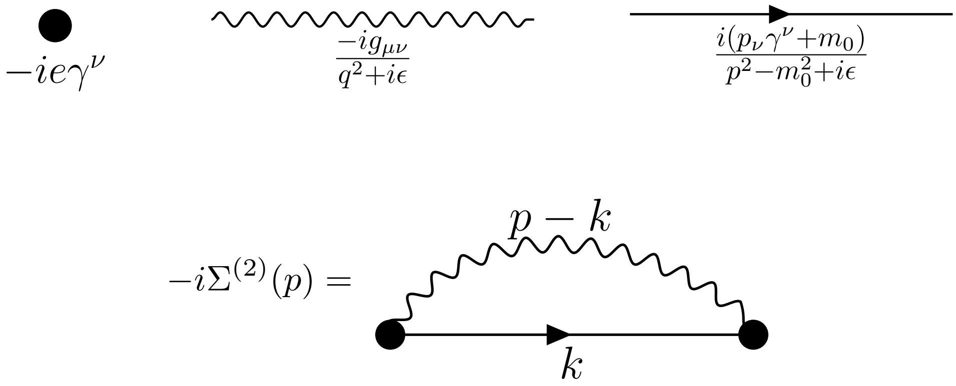

Figure 1: Feynman diagram of the lowest contribution to the electron self energy

b) Calculate the "rest energy" in momentum space,

$$\int d^3\mathbf{r}|E(\mathbf{r})|^2 = \int \frac{d^3\mathbf{q}}{(2\pi)^3}|E(\mathbf{q})|^2. \tag{3}$$

Use that the Fourier transform of the potential is $\phi(\mathbf{q}) = e/q^2$ and introduce the regularization scheme by Pauli and Villars:

$$\frac{1}{q^2} \to \left[\frac{1}{q^2} - \frac{1}{q^2+\Lambda^2}\right], \tag{4}$$

where $\Lambda \to \infty$. Compare the result to **a**).

Remark: Besides the UV (small distance/large momenta) divergence discussed above, there is also an IR (large distance/small momenta) divergence in this problem. Technically, since $|\phi(\mathbf{r})|$ is not integrable over $\mathcal{R}^3$, we cannot calculate the Fourier transform directly. Instead, we can use an exponential cutoff at large r as the Yukawa potential $\phi_Y(\mathbf{r}) = e^{-\mu r}\phi(\mathbf{r})$ with the Fourier transform $\phi_Y(\mathbf{q}) \sim 1/(q^2+\mu^2)$.

3. The Mass shift of electrons in QED (*)

The mass shift of the spin up electron is given by its self energy:

$$\delta m = m - m_0 \simeq \Sigma_{11}(p_\nu\gamma^\nu = m_0), \tag{5}$$

for $\mathbf{p} = (m_0, 0, 0, 0)^T$. Here, m_0 is the bare mass of the electron. We shall approximate the self energy in the lowest order,

$$\Sigma(p_\nu\gamma^\nu = m_0) \simeq \Sigma^{(2)}(p_\nu\gamma^\nu = m_0), \tag{6}$$

where $\Sigma^{(2)}$ (the second order in e) is defined by the Feynmann-diagram in Fig. 1.

a) Using the Feynman rules of QED, write down the expression for $-i\Sigma^{(2)}(p)$.

b) Show that $\gamma^\mu\gamma_\mu = 4$. Use this relation and the anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$ for the Dirac matrices to simplify the numerator.

c) As discussed in the problem **2.**, we need to regularize the photon propagator. Do this in two steps: i) insert a small photon "mass" μ (IR divergence) and ii) use the Pauli-Villars procedure with a large Λ (UV divergence):

$$\frac{1}{q^2 + i\epsilon} \to \frac{1}{q^2 - \mu^2 + i\epsilon} \to \frac{1}{q^2 - \mu^2 + i\epsilon} - \frac{1}{q^2 - \Lambda^2 + i\epsilon}. \tag{7}$$

Introduce a "Feynmann parameter" x to merge the denominators, using the identity

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}. \tag{8}$$

(It is useful to substitute the electron propagator as "$1/B$"). Shift the momentum integration to $l^\nu = k^\nu - xp^\nu$ by completing the square in the denominator.

d) Perform the Wick rotation, by introducing $l_E^0 = -il^0$. Calculate the integral over l_E to show that

$$\Sigma^{(2)}(p) = \frac{e^2}{8\pi^2} \int_0^1 dx(2m_0 - xp_\nu\gamma^\nu) \log\left(\frac{x\Lambda^2}{-x(1-x)p^2 + x\mu^2 + (1-x)m_0^2}\right). \tag{9}$$

Here the integral measure is $d^4 l_E = 2\pi^2 l_E^3 dl_E$.

e) Take $\Sigma_{11}^{(2)}(\mathbf{p})$ with momentum $\mathbf{p} = (m_0, 0, 0, 0)^T$ to get the lowest order contribution to the mass shift of the spin up electron and show that

$$\delta m \simeq \Sigma_{11}^{(2)}(p_\nu\gamma^\nu = m_0) = \frac{e^2 m_0}{8\pi^2} \int_0^1 dx(2-x) \log\left(\frac{x\Lambda^2}{x\mu^2 + m_0^2(1-x)^2}\right). \tag{10}$$

Check either numerically or analytically that one can approximate the argument of the logarithm by $(\Lambda/m_0)^2$ in the limit $\mu \ll m_0 \ll \Lambda$. Then it is easy to show the final result is

$$\delta m \to \frac{3m_0 e^2}{16\pi^2} \log\left(\frac{\Lambda^2}{m_0^2}\right). \tag{11}$$

The mass shift is ultraviolet divergent.