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Exercise sheet online at http://www.thp.uni-koeln.de/~jinhong/qft2

Quantum Field Theory II
Problem Set 3

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

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1. Non-abelian gauge fields (*)

The non-commutative generalization of the single-component Dirac gauge theory to $\mathfrak{su}(N)$ involves a N component vector of spinors $\psi = (\psi_1, ..., \psi_N)$. The former gauge field A_μ needs to be replaced by a traceless $N \times N$-matrix G_μ. While a gauge transformation acting on the field ψ involves a matrix multiplication,

$$\psi \longrightarrow U(x)\psi,$$

G_μ transforms according to

$$G_\mu \longrightarrow U(x)G_\mu U^\dagger(x) - \frac{i}{g_s}U(x)\partial_\mu U^\dagger(x).$$

a) In the case of $\mathfrak{u}(1)$, recover the familiar transformation laws $\psi \to e^{i\phi}\psi, \quad A_\mu \to A_\mu - \frac{1}{q_s}\partial_\mu\phi$ from the generalized formula.

b) Show that a Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + ig_s G_\mu)\psi$$

is gauge invariant. We use the relation of $\partial_\mu \left(U^\dagger U\right) = 0$.

c) Show that the field strength tensor

$$G_{\lambda\rho} := \partial_\lambda G_\rho - \partial_\rho G_\lambda + ig_s[G_\lambda, G_\rho]$$

transform as

$$G_{\lambda\rho} \longrightarrow U(x)G_{\lambda\rho}U^\dagger(x), \tag{1}$$

such that $\mathrm{Tr}\left[G_{\lambda\rho}G^{\lambda\rho}\right]$ is gauge invariant.

2. $\mathfrak{su}(3)$ Hubbard-model

The $\mathfrak{su}(3)$ Hubbard-Hamiltonian for fermion $c_{\alpha,i}$ has the form

$$H = -t\sum_{<ij>}\sum_{\alpha=1}^{3}(c^\dagger_{\alpha i}c_{\alpha j} + h.c.) + U\sum_i n_i^2,$$

where i, j are nearest neighbour site-indices and α is a new 3-component internal *color* degree of freedom that replaces the spin 1/2 in the usual Fermi-Hubbard model.

a) Show that the occupation number operator $n_i = \sum_\alpha c^\dagger_{\alpha i} c_{\alpha i}$ and the operator $\frac{1}{6}\epsilon_{\alpha\beta\gamma} c^\dagger_\alpha c^\dagger_\beta c^\dagger_\gamma$ are invariant under $SU(3)$.

b) For attractive interaction and zero hopping, $t \to 0,\ U < 0$, show that the ground state of H is a color singlet, i.e. it is invariant under global $SU(3)$ rotations. We assume that the number of fermions in each site is 3.

Hint: For a 3×3 matrix, $\det U$ is given as $\det(U) = \sum_{ijk} \epsilon_{ijk} U_{1i} U_{2j} U_{3k}$. Use this relation to show that $\epsilon_{i'j'k'} \det(U) = \sum_{ijk} \epsilon_{ijk} U_{i'i} U_{j'j} U_{k'k}$.

3. $\mathfrak{su}(3)$ representation theory

The Lie algebra $\mathfrak{su}(3)$ can be represented by the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The matrices $T_i = \frac{\lambda_i}{2}$ span the Lie algebra $\mathfrak{su}(3)$. The two matrices T_3 and T_8 span a maximally commutative subalgebra of $\mathfrak{su}(3)$, which in mathematical terms is called the *Cartan subalgebra.* The eigenvalues of a state with respect to the Cartan subalgebra are its quantum numbers. This is analogous to the case of $\mathfrak{su}(2)$, where the Cartan subalgebra is spanned by σ_z only.

a) The generalized ladder-operators $L_{a,b}$ shall be defined as to raise the eigenvalue of T_3 by the amount a and the eigenvalue of T_8 by b. Show that this is accomplished if

$$[L_{a,b}, T_3] = -a L_{a,b}, \quad [L_{a,b}, T_8] = -b L_{a,b}. \tag{2}$$

b) Show, that

$$L_{\pm 1,0} = \frac{1}{\sqrt{2}}(T_1 \pm i T_2),$$
$$L_{\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}(T_4 \pm i T_5),$$
$$L_{\mp\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}}(T_6 \pm i T_7).$$

Trivially, two more operators can be defined as $L^1_{0,0} = T_3$, $L^2_{0,0} = T_8$, which do not change the quantum numbers.

The $\mathfrak{su}(3)$ algebra can be used to classify bound states of up-, down- and strange quarks (which have rather similar mass). Here one uses the same technique which is used to classify bound states made from spin-1/2 objects, which are classified using $\mathfrak{su}(2)$ representations (using e.g. singlets or triplets). As discussed above, $\mathfrak{su}(3)$ has one more quantum number compared to $\mathfrak{su}(2)$. For example there is a 8-dimensional representation where the quantum numbers are given by the values of a and b computed above. One can group certain baryon states (bound states of 3 quarks) with these quantum numbers into octets. This means that there are 8 baryons with quantum numbers (a, b) which have approximately the same mass.