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Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 4

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. Lattice gauge theory (*)

Let us consider a two-dimensional lattice gauge model described by the Hamiltonain

$$H = -\sum_{\langle i,j\rangle} u_{ij} c_i^\dagger c_j. \tag{1}$$

Here c_i is the operator to annihilate a fermion residing in site $\mathbf{r}_i = (x, y)$ and the $\langle i, j\rangle$ represent summation on nearest neighbour pairs. u_{ij} is written in terms of a vector potential $\mathbf{A}$ as

$$u_{ij} = \exp\left(i\int_{\mathbf{r}_j}^{\mathbf{r}_i} \mathbf{A}\cdot d\mathbf{r}\right). \tag{2}$$

a) Show that the Hamiltonian is invariant under the local $U(1)$ gauge transformations of

$$c_i \to e^{i\phi(\mathbf{r}_i)} c_i \qquad \mathbf{A} \to \mathbf{A} + \nabla\phi(\mathbf{r}_i). \tag{3}$$

b) Let us define an operator

$$U(\Box) = (u_{i,i+a\mathbf{y}})\,(u_{i+a\mathbf{y},i+a\mathbf{x}+a\mathbf{y}})\,(u_{i+a\mathbf{x}+a\mathbf{y},i+a\mathbf{x}})\,(u_{i+a\mathbf{x},i})\,. \tag{4}$$

for an elementary plaquette $\Box$ in lattice unit directions $\mathbf{x}$ and $\mathbf{y}$. Here a is the lattice constant. Show that the Hamiltonian

$$H_g = -g\sum_{\Box} U(\Box) + \text{h.c.} \tag{5}$$

is gauge invariant. Here $\sum_\Box$ is the sum over all elementary plaquettes. Rewrite $U(\Box)$ in terms of A and expand for small area a^2. Show that

$$H_g = \text{const.} + ga^2\int d^2\mathbf{r} B_z^2(\mathbf{r}). \tag{6}$$

c) Let us generalize it to the $SU(N)$ case. Then, c_i becomes an N component fermion field $c_i = (c_{i,\alpha})$ where $i = 1, ..., N$, and u_{ij} becomes $SU(N)$ rotation matrix. Show that the bare Hamilonian (Eq. (1)) and $\mathrm{Tr}\,[U(\square)]$ are gauge invariant under the gauge transformation

$$u_{ij} \to U(\mathbf{r}_i) u_{ij} U^\dagger(\mathbf{r}_j), \qquad c_i \to U(\mathbf{r}_i) c_i. \tag{7}$$

2. Bound states with linear potential

Confinement leads to a linear potential between particles. Here we want to estimate the properties of the bound states. We first consider two massive particles with mass m in the non-relativistic limit, where the Schrödinger equation for the relative coordinate is given by

$$-\frac{\hbar^2}{2\mu}\partial_x^2 \psi(x) + \lambda |x| \psi(x) = E\psi(x), \tag{8}$$

with the effective mass $\mu = m/2$.

a) For given size L of the bound states, the kinetic energy cost is estimated as $\hbar^2/(2\mu L^2)$ and the potential energy estimated as λL. Estimate the magnitude of the binding energy and L minimizing the total energy.

A side remark: The exact solutions of Eq. (8) is given as

$$E_n = z_n \left(\frac{\hbar^2 \lambda^2}{\mu}\right)^{1/3}, \tag{9}$$

with integer $n = 1, 2, 3....$ The z_n's are the negative zeros of the Airy function of the first kind with $z_1 = 2.338$.

b) Can you make estimate the binding energy for two massless particles? Estimate for which value of λ one can use the non-relativistic approximation. Note that about 99% of the mass of proton is from the binding energy.

3. Confinement in a 1+1 dimensional system

In 1+1 dimensions, a gauge field A_μ ($\mu = 0, 1$) is coupled to a charge matter with current density j^μ. The action is written as

$$S = \int dx dt \left(-\frac{F_{01}F^{01}}{2} + j^\mu A_\mu \right). \tag{10}$$

The corresponding field strength has just a single component F_{01}, the electric field in x direction.

a) Show that the equation of motion for A_0 leads to

$$j^0 = \partial_x F^{01}. \tag{11}$$

b) Consider a point charge e at position $x = -L/2$ and a charge $-e$ at position $x = L/2$. Then the charge density is written as $j^0 = e\,[\delta(x + L/2) - \delta(x - L/2)]$. Solve Eq. (11) with a boundary condition that the electric field vanishes at $x \to \pm\infty$, obtaining

$$
\begin{aligned}
F^{01} &= e \quad (-L/2 < x < L/2) \\
&= 0 \quad (|x| > L/2).
\end{aligned}
\tag{12}
$$

c) Show that the energy stored in the electric field

$$
H = \frac{1}{2} \int dx \left(F^{01}\right)^2 \tag{13}
$$

is linear in the separation L. Electrical charges in 1+1 dimensions are classically confined. Actually, we have seen this phenomena in a parallel plate capacitor where the electric field between the plates is constant and thus the stored energy is linear in the distance between the plates.