Institute of Theoretical Physics — A. Rosch
University of Cologne — L. Heinen, V. Lohani, H. F. Legg, J. Park
Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

# Quantum Field Theory II
# Problem Set 6

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

## 1. An instructive integral

Calculate the integral

$$I(\alpha) = \int_{-\infty}^{\infty} e^{\alpha(x^2-x^4)} dx$$

for large positive $\alpha$ up to the next to leading order. Compare your result with numerical values for $I(1)$ and $I(10)$.

## 2. Central limit theorem (*)

The central limit theorem states that the mean of a sufficiently large number of independent and identically distributed random variables (not necessarily gaussian distributed), each with finite mean and variance, will be approximately Gaussian distributed. The aim of this exercise is to prove the theorem by means of a saddle-point approximation.

**a)** The goal will be the probability distribution for the random variable $X = \frac{1}{N}\sum_{i=1}^{N} x_i$ in the limit $N \to \infty$, where the $x_i$ are distributed according to the probability distribution $p(x_i)$. Thus, the quantity of interest is

$$P(X) = \int \delta(X - \frac{1}{N}\sum_i x_i) \prod_i p(x_i) dx_i$$

To proceed further, use the integral representation of the delta function,

$$\delta(x) = \int \frac{d\lambda}{(2\pi)} e^{i\lambda x}$$

Denoting $e^{-S[\alpha]} = \prod_i \int dx_i p(x_i) e^{-i\alpha x_i}$, write $P(X)$ as an integral over $\lambda$ only. Which $\lambda$ mostly contributes to the integral? What is the saddle point equation? (You do not need the explicit solution.)

**b)** Expand the exponent of the integrand to quadratic order around $\lambda = 0$ and perform the integral over $\lambda$ to obtain the central limit theorem. What are the mean and variance of $X$?

**c)** Why does for this problem the expansion around $\lambda = 0$ approximately give the same answer as the expansion around the saddle point? Why does this approximation become exact in the limit $N \to \infty$?

## 3. Hubbard-Stratonovich transformation

The aim of this exercise is to derive the Hubbard-Stratonovich action using two different ways. The Hubbard model for spinful fermions with on-site Coulomb interaction is given by

$$H = -\sum_{<i,j>}\sum_{\alpha=\uparrow,\downarrow} t_{ij} c^\dagger_{i\alpha} c_{j\alpha} + U\sum_i n_{i\uparrow} n_{i\downarrow} - \mu_0 \sum_i \sum_\alpha n_{i\alpha}. \tag{1}$$

Here the number operator is defined by $n_{i\alpha} \equiv c^\dagger_{i\alpha} c_{i\alpha}$ and the $< i, j >$ represents summation on nearest neighbour pairs.

**a)** Derive this useful relation of

$$n_{i,\uparrow} n_{i,\downarrow} = -\frac{2}{3}\boldsymbol{S}_i^2 + \frac{1}{2}(n_{i,\uparrow} + n_{i,\downarrow}), \tag{2}$$

where the spin operators $\boldsymbol{S}_i$ are written as $\boldsymbol{S}_i = \frac{1}{2}\sum_{\alpha\alpha'} c^\dagger_{i\alpha} \boldsymbol{\sigma}_{\alpha\alpha'} c_{i\alpha'}$. You may use the identity

$$\sum_{a=1,2,3} \sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}. \tag{3}$$

**b)** Using the Hubbard-Stratonovich transformation with a real vector field $\boldsymbol{\phi}$ and Eq. (2), derive the effective action

$$S_{\mathrm{HS}}[c, \boldsymbol{\phi}] = \int_0^\beta d\tau \left( \sum_{ij} c^*_{i\alpha} \left( \left( (\partial_\tau - \mu)\,\delta_{\alpha\alpha'} + \boldsymbol{\phi}_i(\tau) \cdot \frac{\boldsymbol{\sigma}_{\alpha\alpha'}}{2} \right) \delta_{ij} - t_{ij}\delta_{\alpha\alpha'} \right) c_{j\alpha'} + \frac{3}{8U}\sum_i \boldsymbol{\phi}_i(\tau)^2 \right). \tag{4}$$

Note that the term of $U(n_{i,\uparrow} + n_{i,\downarrow})/2$ just changes the chemical potential as $\mu = \mu_0 - U/2$.

**c)** Apply the Hubbard-Stratonovic transformation with a complex field $\phi$ directly to $Un_\uparrow n_\downarrow$. For the form $e^{A^* V B}$ in the lecture, identify $A^* = n_\uparrow$ and $B = n_\downarrow$. Write the effective action in terms of $\phi_1$, $\phi_2$ with $\phi = \phi_1 + i\phi_2$. To what quantities do $\phi_1$ and $\phi_2$ couple?