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Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 7

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. Magnetism in the Hubbard model: Spin-waves (*)

On the last exercise sheet, the following effective action for a three-dimensional Hubbard model was derived:

$$S_{\mathrm{HS}}[c, c^*, \boldsymbol{\phi}] = \int_0^\beta d\tau \left(\sum_{ij} c^*_{i\alpha} \left(\left((\partial_\tau - \mu)\, \delta_{\alpha\alpha'} + \boldsymbol{\phi}_i(\tau) \cdot \frac{\boldsymbol{\sigma}_{\alpha\alpha'}}{2} \right) \delta_{ij} - t_{ij}\delta_{\alpha\alpha'} \right) c_{j\alpha'} + \frac{1}{2J} \sum_i \boldsymbol{\phi}_i(\tau)^2 \right). \tag{1}$$

Here c^* and c are Grassmann numbers to describe the local electron creation and annihilation, ϕ is a three-component real Hubbard-Stratonovich field, and $J = 4U/3$ is related to the on-site interaction strength.

a) Integrating out c and c^* to arrive at the expression

$$S[\phi] = \int_0^\beta d\tau \left(\frac{1}{2J} \sum_i \boldsymbol{\phi}_i(\tau)^2 \right) - \mathrm{Tr}\log(-G^{-1}).$$

Here G^{-1} is a matrix. What is the form of the matrix (in imaginary time space)?

b) For the *ferromagnetic solution* of the Hubbard model, we will assume a spatially homogeneous and τ-independent mean-field and make the ansatz,

$$\boldsymbol{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ \phi_0 \end{pmatrix}$$

Derive the saddle-point equation determining ϕ_0. Derive the expression of the Green's function G in the momentum and the frequency space.

c) Assuming that the dominant contribution to the action comes from the mean-field and its Gaussian fluctuations, make the substitution

$$\boldsymbol{\phi} \approx \boldsymbol{\phi}_0 + \boldsymbol{\delta\phi} = \boldsymbol{\phi}_0 + \begin{pmatrix} \delta\phi_x \\ \delta\phi_y \\ \delta\phi_z \end{pmatrix}$$

and expand the action to second order in $\boldsymbol{\delta\phi}$. Show that the action for $\boldsymbol{\delta\phi}$ has the form of

$$S[\boldsymbol{\delta\phi}] = \sum_{\ell,\ell'=x,y,z} \sum_{\Omega_n} \sum_{\boldsymbol{p}} \delta\phi_\ell(i\Omega_n, \boldsymbol{p}) \chi^{-1}_{\ell\ell'}(i\Omega_n, \boldsymbol{p}) \delta\phi_{\ell'}(-i\Omega_n, -\boldsymbol{p}), \tag{2}$$

with bosonic Matsubara frequencies $\Omega_n = 2\pi n/\beta$.

d) Show that the retarded correlation functions $(\chi_R^{-1})_{zz}$, $(\chi_R^{-1})_{xx}$, and $(\chi_R^{-1})_{xy}$ are written as

$$\begin{aligned}(\chi_R^{-1})_{zz}(\Omega, \boldsymbol{p}) = \frac{1}{2J} + \frac{1}{4} \sum_{\boldsymbol{k}} \frac{1}{\Omega - \epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \epsilon_{\boldsymbol{k}} + i\eta} \Big(& n_F(\epsilon_{\boldsymbol{k}} - \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} - \phi_0/2) \\ & + n_F(\epsilon_{\boldsymbol{k}} + \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \phi_0/2) \Big),\end{aligned} \tag{3}$$

$$\begin{aligned}(\chi_R^{-1})_{xx}(\Omega, \boldsymbol{p}) = \frac{1}{2J} + \frac{1}{4} \sum_{\boldsymbol{k}} \Bigg(& \frac{1}{\Omega - \epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \epsilon_{\boldsymbol{k}} - \phi_0 + i\eta} \left(n_F(\epsilon_{\boldsymbol{k}} - \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \phi_0/2) \right) \\ & + \frac{1}{\Omega - \epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \epsilon_{\boldsymbol{k}} + \phi_0 + i\eta} \left(n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} - \phi_0/2) \right) \Bigg),\end{aligned} \tag{4}$$

$$\begin{aligned}(\chi_R^{-1})_{xy}(\Omega, \boldsymbol{p}) = -\frac{i}{4} \sum_{\boldsymbol{k}} \Bigg(& \frac{1}{\Omega - \epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \epsilon_{\boldsymbol{k}} - \phi_0 + i\eta} \left(n_F(\epsilon_{\boldsymbol{k}} - \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \phi_0/2) \right) \\ & - \frac{1}{\Omega - \epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \epsilon_{\boldsymbol{k}} + \phi_0 + i\eta} \left(n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} + \phi_0/2) - n_F(\epsilon_{\boldsymbol{k}+\boldsymbol{p}} - \phi_0/2) \right) \Bigg).\end{aligned} \tag{5}$$

Here $\epsilon_{\boldsymbol{k}} = -2t(\cos(k_x a) + \cos(k_y a) + \cos(k_z a)) - \mu$ is the energy of a bare electron with momentum $\boldsymbol{k}$. You may use a nice sum rule

$$\frac{1}{\beta} \sum_{\omega_n} \frac{1}{i\omega_n - \epsilon} = n_F(\epsilon), \tag{6}$$

for fermionic Matsubara frequencies ω_n. The retarded correlation functions are obtained from analytic continuation $(\chi_R^{-1})_{\ell\ell'} = \chi^{-1}_{\ell\ell'}(i\Omega \to \Omega + i\eta)$.

e) Show that in the limit of $\Omega \to 0$ and $\boldsymbol{p} \to 0$, $(\chi_R^{-1})_{xx}$ and $(\chi_R^{-1})_{xy}$ go to zero. Use the saddle point equation for ϕ_0. Also show that in the limit, $(\chi_R^{-1})_{zz}$ becomes finite. Note that it implies that the longitudinal fluctuation is massive while the transverse fluctuation (i.e. goldstone mode) is massless.

f) The physical interpretation of the fluctuations on top of the mean-field solution is *spin-waves*. Expand $(\chi_R^{-1})_{xx}$ and $(\chi_R^{-1})_{xy}$ in small $\boldsymbol{p}$ and Ω and show that χ_R^{-1} for x and y directions has the form of

$$\chi_R^{-1} \equiv \begin{pmatrix} (\chi_R^{-1})_{xx} & (\chi_R^{-1})_{xy} \\ (\chi_R^{-1})_{yx} & (\chi_R^{-1})_{yy} \end{pmatrix} \simeq \begin{pmatrix} a\boldsymbol{p}^2 & ib\Omega \\ -ib\Omega & a\boldsymbol{p}^2 \end{pmatrix}, \tag{7}$$

with constants a and b. Notice that χ_R has a pole structure at $\Omega = \pm\sqrt{\frac{a}{b}}\boldsymbol{p}^2$ to reflect the dispersion relation of the goldstone mode.