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Exercise sheet online at http://www.thp.uni-koeln.de/∼jinhong/qft2

Quantum Field Theory II
Problem Set 8

Please hand in your solution of the exercise marked with (*), for which we offer correction, in the mail box until Monday 12am.

1. Quantum phase transition: self-consistent one-loop approximation (*)

The goal of this exercise is to draw a phase diagram as a function of temperature T and a parameter (r_0 in this exercise) under the self-consistent Hartree-Fock approximation. To this end, we consider an action for a single-component bosonic field $\phi_{\mathbf{k}}(i\omega_n)$, written as

$$S = \sum_{\mathbf{k},\omega_n} \phi^*_{\mathbf{k}}(i\omega_n)(r_0 + \mathbf{k}^2 + |\omega_n|^{\frac{2}{z}})\phi_{\mathbf{k}}(i\omega_n) + \frac{g}{4\beta\mathcal{V}} \sum_{\omega_{n_1},\omega_{n_2},\omega_{n_3}} \sum_{\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}} \phi^*_{\mathbf{k_1}}(i\omega_{n_1})\phi^*_{\mathbf{k_2}}(i\omega_{n_2})\phi_{\mathbf{k_3}}(i\omega_{n_3})\phi_{\mathbf{k_1}+\mathbf{k_2}-\mathbf{k_3}}(i\omega_{n_1} + i\omega_{n_2} - i\omega_{n_3}). \quad (1)$$

Here z is called "dynamical critical exponent" with, e.g., $z = 1$ for an antiferromagnetic transition in an insulator while $z = 2$ in a metal due to the friction from electrons (Landau-damping). In small g limit, the first-order perturbation is taken. Taking into account the Hartree-Fock diagram, the full Green's function for the field is given by

$$G_{\mathbf{k}}(i\omega_n) \equiv \langle \phi^*(\mathbf{k}, i\omega_n)\phi(\mathbf{k}, i\omega_n)\rangle = \frac{1}{r_0 + \mathbf{k}^2 + |\omega_n|^{\frac{2}{z}} - \Sigma} = \frac{1}{r + \mathbf{k}^2 + |\omega_n|^{\frac{2}{z}}}, \quad (2)$$

where the Hartree-Fock contribution to the self energy is just a number independent of ω_n and $\boldsymbol{k}$ given by

$$\Sigma = -\frac{g}{\beta\mathcal{V}} \sum_{i\omega_n,\mathbf{k}} G_{\mathbf{k}}(i\omega_n) \equiv \delta r(r, T), \quad (3)$$

and redefine a parameter $r \equiv r_0 - \delta r$. r^c_0 is defined as $r^c_0 \equiv \delta r(r = 0, T = 0)$. With this definition, r vanishes for $r_0 = r^c_0$ at $T = 0$.

a) Let us consider the zero temperature case. How does r depend on $r_0 - r_0^c$? At zero temperature, the summation $\frac{1}{\beta}\sum_{i\omega_n}$ becomes the integral form of $\int (d\omega/2\pi)$. You may use the identity

$$\int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{1}{r + k^2 + |\omega|^{2/z}} = D \int d^{d+z}k \frac{1}{r + k^2}. \tag{4}$$

D is a numerical constant. This identity implies that a quantum theory in d dimension is related to a classical theory in $d + z$ dimension. The integral of the right hand side of Eq. (4) is approximately computed by the division of the integral

$$\int_0^{k_{\max}} dk = \int_0^{\sqrt{r}} dk + \int_{\sqrt{r}}^{k_{\max}} dk, \tag{5}$$

and setting the denominator

$$r + k^2 \simeq \begin{cases} r, & r > k^2 \\ k^2, & r < k^2 \end{cases}. \tag{6}$$

Show that it leads to the self-consistent equation

$$r = r_0 - r_0^c + C r^{\frac{d+z-2}{2}} \qquad \text{for } d + z > 2. \tag{7}$$

C is a numerical constant and you do not need to compute C. The condition $d + z > 2$ is needed to avoid the singularity at $r = 0$.

b) Now we are interested in the static susceptibility $\chi(\boldsymbol{k} = 0, \omega = 0) = 1/r$. Show that depending on the condition of the exponent $d + z$, $\chi(\boldsymbol{k} = 0, \omega = 0)$ in small r limit has different scalings in $1/(r_0^c - r_0)$. What is the scaling of $\chi(\boldsymbol{k} = 0, \omega = 0)$ for two different cases $d + z > 4$ and $2 < d + z < 4$?

c) The correlation length ξ is determined by $\xi = 1/\sqrt{r}$ from the right hand side of Eq. (2). Find the scaling exponent ν of ξ in $1/(r_0^c - r_0)$ for the different cases $d + z > 4$ and $2 < d + z < 4$.

d) Properties at $T > 0$ can be obtained from the T dependence of δr. But it is tedious to find the T dependence of δr using the Matsubara summation. Instead, we use a dimensional analysis to know the T dependence of δr. For the case of $d+z > 4$, estimate T_c assuming this T dependence is valid in $T \simeq T_c$. What is the scaling exponent ψ in the formula $T_c \sim (r_0^c - r_0)^{\psi}$? A schematic phase diagram (T_c as a function of r_0) is drawn in Fig. 1.

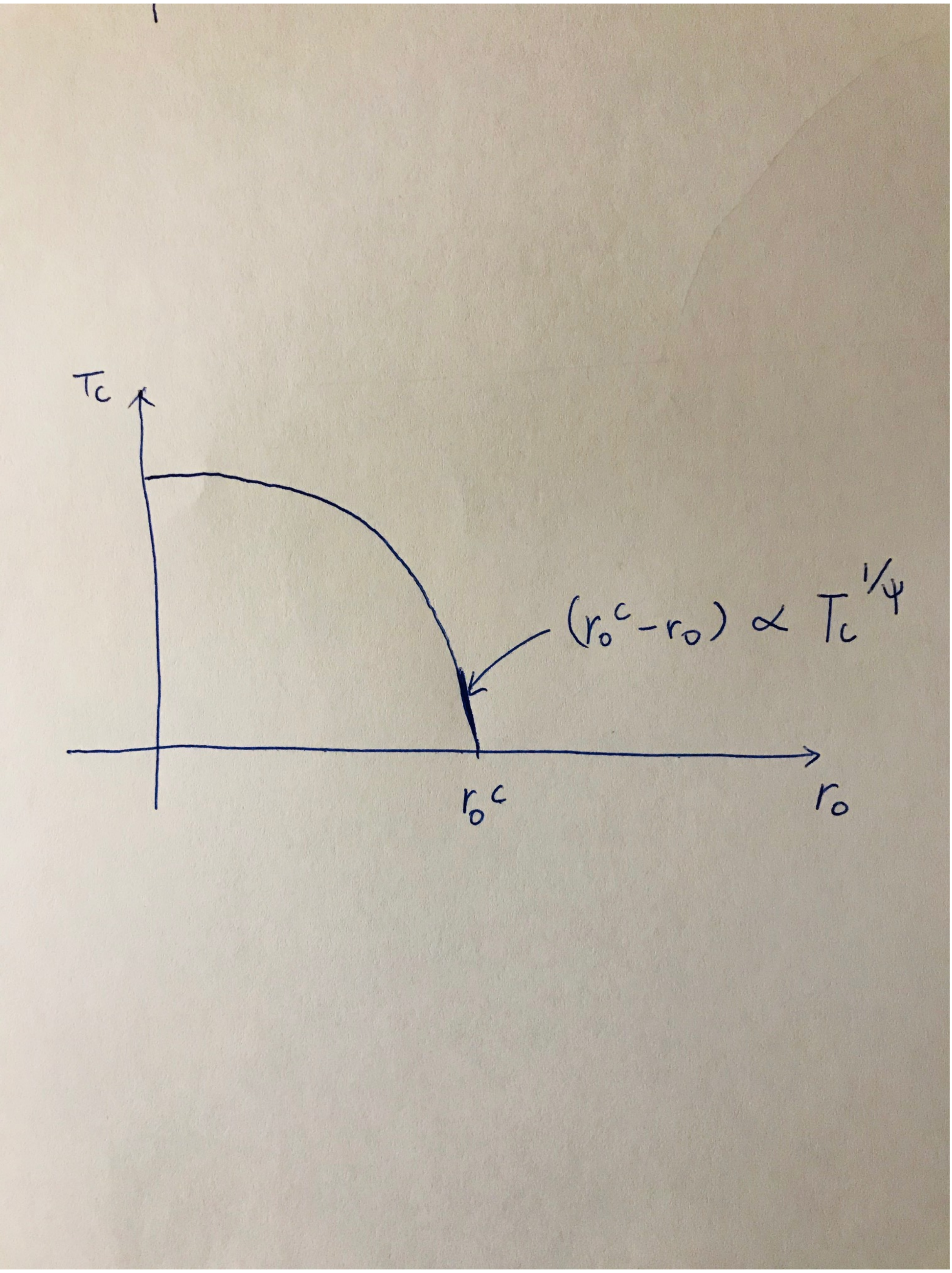

Figure 1: A schematic phase diagram of T_c as a function of r_0.