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Quantum Field Theory II
Solution to Problem Set 6

1. An instructive integral

First determine the saddle points (SPs). The SP-equation reads:

$$\mathrm{e}^{\alpha\left(x^2+x^4\right)} \longrightarrow \frac{\partial}{\partial x}\alpha\left(x^2+x^4\right) = \alpha\left(2x+4x^3\right) \overset{!}{=} 0$$

$$\Rightarrow x = 0 \vee x = \pm\frac{1}{\sqrt{2}}$$

$x = 0$ is a local minimum since $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\alpha\left(x^2+x^4\right)|_{x=0} = 2\alpha > 0$. Thus we have two SPs at $x = \pm\frac{1}{\sqrt{2}}$. Now the integral reads:

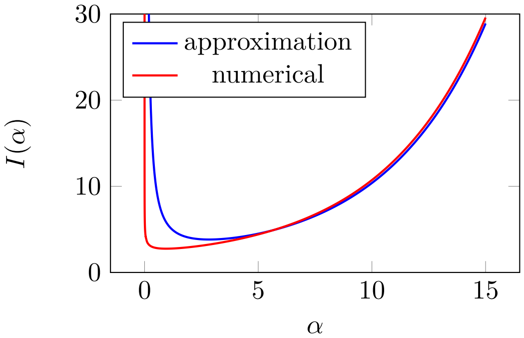
$$\begin{aligned} I(\alpha) &= \int_{-\infty}^{\infty} \mathrm{e}^{\alpha\left(x^2+x^4\right)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha\left(\frac{1}{4}-2\left(x-\frac{1}{\sqrt{2}}\right)^2\sqrt{8}\left(x-\frac{1}{\sqrt{2}}\right)^3-\left(x-\frac{1}{\sqrt{2}}\right)^4\right)}\,\mathrm{d}x \\ &= \frac{\mathrm{e}^{\frac{\alpha}{4}}}{\sqrt{\alpha}}\int_{-\infty}^{\infty} \mathrm{e}^{-2y^2-\sqrt{\frac{8}{\alpha}}y^3-\frac{y^4}{\alpha}}\,\mathrm{d}y\,, \end{aligned}$$

where we substituted $y \coloneqq \sqrt{\alpha}\left(x-\frac{1}{\sqrt{2}}\right) \Rightarrow \mathrm{d}x = \frac{1}{\sqrt{\alpha}}\,\mathrm{d}y$ in the last equality. Up to now everything was exact. Now we will Taylor-expand the integrand around $\frac{1}{\sqrt{\alpha}} \approx 0$, up to order $O\left(\frac{1}{\alpha}\right)$. This implicitly assumes that $|y| \ll \alpha$ (see the integrand). The expansion is thus akin to a SP-approximation (indeed the SP-approximation is the leading term). We therefore have to do the series expansion around each of the SPs and add the integrals. Fortunately, in this case, both are equal and we only get an overall factor of 2.

$$I(\alpha) \approx 2\frac{\mathrm{e}^{\frac{\alpha}{4}}}{\sqrt{\alpha}}\int_{-\infty}^{\infty}\mathrm{e}^{-2y^2}\left(1-\sqrt{\frac{8}{\alpha}}y^3+\frac{1}{\alpha}y^4\left(4y^2-1\right)\right)\mathrm{d}y = \frac{\sqrt{2\pi}\,\mathrm{e}^{\frac{\alpha}{4}}}{\sqrt{\alpha}}\left(1+\frac{3}{4\alpha}\right)$$

The y^3-term vanishes identically, since it is odd in y and integrated over all of $\mathbb{R}$. Note that much of the calculation was only done to extract the dominant $\frac{\mathrm{e}^{\frac{\alpha}{4}}}{\sqrt{\alpha}}$-factor. Before that, expanding around $\frac{1}{\sqrt{\alpha}} \approx 0$ (or $\alpha \approx \infty$) would not have worked.
Now to compare our approximation with numerical results:

α	num.	appr.
1	2.761	5.632
10	10.726	10.380



In the figure we can see that our assumption $|y| \ll \alpha$ brakes down below $\alpha \approx 5$. For lower α, values of y with $|y| \gtrsim \alpha$ become important to the integral, decreasing the accuracy of our approximation. Note that the integral can actually be computed to be exactly

$$I(\alpha) = \frac{\pi}{\sqrt{8}}\,\mathrm{e}^{\alpha/8}\left(I_{\frac{1}{4}}\left(\frac{\alpha}{8}\right)+I_{-\frac{1}{4}}\left(\frac{\alpha}{8}\right)\right),$$

where $I_n(z)$ is the modified Bessel function of the first kind.

2. Central limit theorem

a) Rewrite $P(X)$ by first inserting the integral representation of the delta function.

$$\begin{aligned} P(X) &= \int \delta\left(X - \frac{1}{N}\sum_i x_i\right)\prod_i p\left(x_i\right)\mathrm{d}x_i = \iint \mathrm{e}^{\mathrm{i}\lambda\left(\left(X - \frac{1}{N}\sum_i x_i\right)\right.}\frac{\mathrm{d}\lambda}{2\pi}\prod_i p\left(x_i\right)\mathrm{d}x_i \\ &= \int \mathrm{e}^{\mathrm{i}\lambda X}\underbrace{\prod_i\left(\int \mathrm{e}^{-\mathrm{i}\lambda x_i/N}\,p\left(x_i\right)\mathrm{d}x_i\right)}_{=\mathrm{e}^{-S\left[\frac{\lambda}{N}\right]}}\frac{\mathrm{d}\lambda}{2\pi} = \int \mathrm{e}^{\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right]}\frac{\mathrm{d}\lambda}{2\pi} \end{aligned}$$

For the second line we used the definition of S from the exercise sheet with $\alpha = \frac{\lambda}{N}$. The SP-equation then reads:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right]\right) \stackrel{!}{=} 0,$$

of which the solution λ_s is complex. The largest contribution to the integral then comes from around $\mathrm{Re}\left(\lambda_s\right)$.

b) Expanding the exponent of the integrand around $\lambda = 0$ gives:

$$\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right] = \mathrm{i}\lambda X - S[0] - \frac{\mathrm{d}}{\mathrm{d}\lambda}S\left[\frac{\lambda}{N}\right]\bigg|_{\lambda=0}\lambda - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}S\left[\frac{\lambda}{N}\right]\bigg|_{\lambda=0}\lambda^2 + O\left(\lambda^3\right).$$

Before calculating each of these terms, observe:

$$\begin{aligned} \mathrm{e}^{-S[\alpha]} &= \prod_i\left(\int \mathrm{e}^{-\mathrm{i}\alpha x_i}\,p\left(x_i\right)\mathrm{d}x_i\right) = \left(\int \mathrm{e}^{-\mathrm{i}\alpha x}\,p\left(x\right)\mathrm{d}x\right)^N \\ \mathrm{e}^{-S[0]} &= \left(\int \mathrm{e}^{-\mathrm{i}0x}\,p\left(x\right)\mathrm{d}x\right)^N = \underbrace{\left(\int p\left(x\right)\mathrm{d}x\right)}_{=1\text{ (normalization)}}{}^N = 1 \\ &\Rightarrow S[0] = 0 \end{aligned}$$

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}\lambda}S\left[\frac{\lambda}{N}\right]\bigg|_{\lambda=0} &= \left[\mathrm{e}^{S\left[\frac{\lambda}{N}\right]}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{e}^{-S\left[\frac{\lambda}{N}\right]}\right]_{\lambda=0} \stackrel{\text{def. of }S}{=} \left[\left(\int\dots\right)^{-N}N\left(\int\dots\right)^{N-1}\int -\mathrm{i}\frac{x}{N}\,\mathrm{e}^{-\mathrm{i}\lambda x/N}\,p\left(x\right)\mathrm{d}x\right]_{\lambda=0} \\ &= -\mathrm{i}\frac{\int x\,\mathrm{e}^0\,p\left(x\right)\mathrm{d}x}{\underbrace{\int \mathrm{e}^0\,p\left(x\right)\mathrm{d}x}_{=1}} = -\mathrm{i}\int xp\left(x\right)\mathrm{d}x = -\mathrm{i}\langle x\rangle_{p(x)} =: -\mathrm{i}\mu \\ -\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}S\left[\frac{\lambda}{N}\right]\bigg|_{\lambda=0} &= \left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathrm{e}^{S\left[\frac{\lambda}{N}\right]}\frac{\mathrm{d}}{\mathrm{d}\lambda}\,\mathrm{e}^{-S\left[\frac{\lambda}{N}\right]}\right]_{\lambda=0} = \left[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\int x\,\mathrm{e}^{-\mathrm{i}\lambda x/N}\,p\left(x\right)\mathrm{d}x}{\int \mathrm{e}^{-\mathrm{i}\lambda x/N}\,p\left(x\right)\mathrm{d}x}\right]_{\lambda=0} \\ &= -\frac{1}{N}\left(\frac{\int x^2\,\mathrm{e}^0\,p\left(x\right)\mathrm{d}x}{\int \mathrm{e}^0\,p\left(x\right)\mathrm{d}x} - \frac{\left(\int x\,\mathrm{e}^0\,p\left(x\right)\mathrm{d}x\right)^2}{\left(\int \mathrm{e}^0\,p\left(x\right)\mathrm{d}x\right)^2}\right) = -\frac{\left\langle x^2\right\rangle_{p(x)} - \langle x\rangle_{p(x)}^2}{N} =: -\frac{\sigma^2}{N}, \end{aligned}$$

where μ and σ are the mean and standard deviation of the original distribution $p(x)$ respectively. Thus we get:

$$\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right] \approx \mathrm{i}\lambda X - \mathrm{i}\mu\lambda - \frac{\sigma^2}{2N}\lambda^2$$

Now we can perform the integral over λ to get:

$$P(X) \approx \int \mathrm{e}^{\mathrm{i}\lambda X - \mathrm{i}\mu\lambda - \frac{\sigma^2}{2N}\lambda^2}\frac{\mathrm{d}\lambda}{2\pi} = \sqrt{\frac{N}{2\pi\sigma^2}}\,\mathrm{e}^{-N\frac{(X-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\left(\sigma/\sqrt{N}\right)^2}}\,\mathrm{e}^{-\frac{(X-\mu)^2}{2\left(\sigma/\sqrt{N}\right)^2}}.$$

Comparing to the formula for a Gaussian distribution, we immediately see:

$$\begin{aligned} \langle X\rangle_{P(X)} &= \langle x\rangle_{p(x)} = \mu \\ \sigma_P^2 := \left\langle X^2\right\rangle_{P(X)} - \langle X\rangle_{P(X)}^2 &= \frac{\sigma^2}{N}. \end{aligned}$$

c) To answer the question of why the expansions around the SP and around $\lambda = 0$ give almost the same result, let us first recall the SP-equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right]\right)\bigg|_{\lambda=\lambda_s} \overset{!}{=} 0 \Rightarrow \mathrm{i}X = \frac{\mathrm{d}}{\mathrm{d}\lambda}S\left[\frac{\lambda}{N}\right]\bigg|_{\lambda=\lambda_s} \overset{\text{as in b)}}{=} \mathrm{i}\frac{\int x\,\mathrm{e}^{-\mathrm{i}\lambda_s x/N}\,p(x)\,\mathrm{d}x}{\int \mathrm{e}^{-\mathrm{i}\lambda_s x/N}\,p(x)\,\mathrm{d}x}.$$

Now let us assume that $\frac{\lambda_s}{N} =: \alpha_s \ll 1$ (or rather $|\alpha_s| \ll |\mu + \sigma|$), which we will self-consistently justify below. Then we can expand the exponentials up to second order in α_s to get:

$$\begin{aligned}
X &\approx \frac{\int x\left(1 - \mathrm{i}x\alpha_s - \frac{1}{2}x^2\alpha_s^2\right)p(x)\,\mathrm{d}x}{\int\left(1 - \mathrm{i}x\alpha_s - \frac{1}{2}x^2\alpha_s^2\right)p(x)\,\mathrm{d}x} = \frac{\mu - \mathrm{i}\alpha_s\left\langle x^2\right\rangle - \frac{1}{2}\alpha_s^2\left\langle x^3\right\rangle}{1 - \mathrm{i}\alpha_s\mu - \frac{1}{2}\alpha_s^2\left\langle x^2\right\rangle} \\
&= \mu - \mathrm{i}\big(\left\langle x^2\right\rangle - \mu^2\big)\,\alpha_s + \frac{1}{2}\left(-\left\langle x^3\right\rangle + 3\left\langle x^2\right\rangle\mu - 2\mu^3\right)\alpha_s^2 + O\big(\alpha_s^3\big) \\
&= \mu - imI\sigma^2\alpha_s + O\big(\alpha_s^3\big) \\
\Rightarrow \alpha_s &\approx \mathrm{i}\frac{X-\mu}{\sigma^2} = \mathrm{i}Z\frac{1}{\sigma\sqrt{N}}.
\end{aligned}$$

In the last step we defined the new variable $Z := \frac{X-\mu}{\sigma/\sqrt{N}}$. We see that the SP is approximately purely imaginary, so that $\mathrm{Re}\{\lambda_s\} \approx 0$, which hints at why expanding around $\lambda = 0$ might be a good approximation. Before we go on, though, let us check self-consistently that α_s is actually small. For this we first examine Z. From the central limit theorem we know that $\langle X\rangle_{P(X)} = \mu$ and $\left\langle X^2\right\rangle_{P(X)} - \langle X\rangle_{P(X)}^2 = \frac{\sigma^2}{N}$. Using this it is easy to show that $\mathrm{mean}(Z) = 0$ and $\mathrm{variance}(Z) = 1$. In a random-variable sense Z does therefore not scale with N: $Z \sim N^0$. And consequently:

$$\alpha_s \approx \mathrm{i}Z\frac{1}{\sigma\sqrt{N}} \sim \frac{1}{\sqrt{N}},$$

which justifies our assumption that α_s is small in the large N limit. To elaborate a bit on the scaling behavior, we can take two different views. Above, we implicitly viewed X as a random variable for fixed N. Consequently Z was seen as a random variable of order 1, and α_s as a random variable of order $1/\sqrt{N} \underset{N\to\infty}{\longrightarrow} 0$. An alternative viewpoint would be to consider P a simple function of X, which would imply $\alpha_s = \Theta\left(N^0\right)$. However, the approximations we do to calculate $P(X)$ are actually only justified when X is close to μ, or more accurately $|X-\mu| \lesssim \frac{2\sigma}{\sqrt{N}} \Leftrightarrow |Z| \lesssim 2$. In this regime $\alpha_s \approx \mathrm{i}\underbrace{Z}_{\lesssim 2}/\left(\sigma\sqrt{N}\right) \sim 1/\sqrt{N}$ as before.

Finally we can do a proper expansion around the SP, to show the connection to $\lambda \approx 0$:

$$\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right] \underset{\lambda\approx\lambda_s}{\approx} \left(\mathrm{i}\lambda_s X - S\left[\frac{\lambda_s}{N}\right]\right) + \left(\mathrm{i}XN - S'\left[\frac{\lambda_s}{N}\right]\right)\frac{(\lambda-\lambda_s)}{N} + \frac{1}{2}\left(-S''\left[\frac{\lambda_s}{N}\right]\right)\frac{(\lambda-\lambda_s)^2}{N^2}.$$

Noting that $\frac{\lambda}{N} = \alpha_s$ small, and $\frac{(\lambda-\lambda_s)}{N} = \frac{\lambda}{N} - \alpha_s$, we can continue by expanding the various derivatives of S around $\alpha_s \approx 0$ to second order, which is justified in the large N limit (and this is why we did the analysis above):

$$\begin{aligned}
\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right] \underset{\lambda\approx\lambda_s}{\overset{\alpha_s\ll 1}{\approx}} &\left(\underbrace{-S[0]}_{0} + \underbrace{(\mathrm{i}XN - S'[0])\frac{\lambda}{N}}_{1} - \underbrace{\frac{1}{2}S''[0]\frac{\lambda^2}{N^2}}_{2}\right) \\
&+ \left(\underbrace{\mathrm{i}XN - S'[0]}_{0} - \underbrace{\mathrm{i}XN + S'[0] - S''[0]\frac{\lambda}{N}}_{1} + \underbrace{S''[0]\frac{\lambda}{N} - \frac{1}{2}S'''[0]\frac{\lambda^2}{N^2}}_{2}\right)\alpha_s \\
&+ \frac{1}{2}\left(\underbrace{-S''[0]}_{0} + \underbrace{2S''[0] - S'''[0]\frac{\lambda}{N}}_{1} - \underbrace{S''[0] + 2S'''[0]\frac{\lambda}{N} - \frac{1}{2}S''''[0]\frac{\lambda^2}{N^2}}_{2}\right)\alpha_s^2.
\end{aligned}$$

The numbers below the braces mark which order of $\frac{(\lambda-\lambda_s)}{N}$ the various terms correspond to. It is immediately obvious that most terms cancel, and indeed, assuming that S is infinitely differentiable (an assumption that is actually very restrictive on the original distribution $p(x)$), one can show that all terms beyond zeroth order in α_s cancel. Finally, by dropping all orders of α_s we get

$$\mathrm{i}\lambda X - S\left[\frac{\lambda}{N}\right] \underset{\lambda\approx\lambda_s}{\overset{\alpha_s\ll 1}{\approx}} -S[0] + \mathrm{i}\lambda X - S'[0]\frac{\lambda}{N} - \frac{1}{2}S''[0]\frac{\lambda^2}{N^2},$$

which is just the same as an expansion around $\lambda \approx 0$. So to answer the original question, the two expansions give approximately the same result because α_s is small, and the approximation becomes exact in the limit $N \to \infty$, because then $\alpha_s \to 0$ as per our analysis above.

3. Hubbard-Stratonovich transformation

a) Starting with the right-hand side of the desired relation and expanding, we get (sums over Greek indices sum over $\{\uparrow, \downarrow\}$, sums over i or j sum over sites, sums over a or b sum over components):

$$\begin{aligned}
-\frac{2}{3}\mathbf{S}_i^2 + \frac{1}{2}\left(n_{i,\uparrow} + n_{i,\downarrow}\right) &= -\frac{1}{6}\sum_{a=1}^{3}\sum_{\alpha\beta\gamma\delta} c_{i,\alpha}^\dagger \sigma_{\alpha\beta}^a c_{i,\beta} c_{i,\gamma}^\dagger \sigma_{\gamma\delta}^a c_{i,\delta} + \frac{1}{2}\sum_\alpha n_{i,\alpha} \\
&= -\frac{1}{6}\sum_{\alpha\beta\gamma\delta}\left(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}\right) c_{i,\alpha}^\dagger c_{i,\beta} c_{i,\gamma}^\dagger c_{i,\delta} + \frac{1}{2}\sum_\alpha n_{i,\alpha} \\
&= -\frac{1}{3}\sum_{\alpha\beta} c_{i,\alpha}^\dagger c_{i,\beta} c_{i,\beta}^\dagger c_{i,\alpha} + \frac{1}{6}\sum_{\alpha\gamma} c_{i,\alpha}^\dagger c_{i,\alpha} c_{i,\gamma}^\dagger c_{i,\gamma} + \frac{1}{2}\sum_\alpha n_{i,\alpha} \\
&= -\frac{1}{3}\sum_{\alpha\beta} c_{i,\alpha}^\dagger \left(-c_{i,\beta}^\dagger c_{i,\beta} + 1\right) c_{i,\alpha} + \frac{1}{6}\sum_{\alpha\beta} n_{i,\alpha} n_{i,\beta} + \frac{1}{2}\sum_\alpha n_{i,\alpha} \\
&= \left(\frac{1}{2} - \frac{2}{3}\right)\sum_\alpha n_{i,\alpha} + \frac{1}{3}\sum_{\alpha\neq\beta} c_{i,\alpha}^\dagger c_{i,\beta}^\dagger c_{i,\beta} c_{i,\alpha} + \frac{1}{6}\sum_{\alpha\neq\beta} n_{i,\alpha} n_{i,\beta} + \frac{1}{6}\sum_\alpha n_{i,\alpha} n_{i,\alpha} \\
&= \left(\frac{1}{3} + \frac{1}{6}\right)\sum_{\alpha\neq\beta} n_{i,\alpha} n_{i,\beta} \\
&= \frac{1}{2}\left(n_{i,\uparrow} n_{i,\downarrow} + n_{i,\downarrow} n_{i,\uparrow}\right) \\
&= n_{i,\uparrow} n_{i,\downarrow}
\end{aligned}$$

Notes:

line 1: insert definition of $\mathbf{S}$

line 2: use equation (3) of the exercise sheet

line 3: expand and execute two sums each

line 4: reorder first term, use definition of n

line 5: reorder terms, note that the now second term is 0 if $\alpha = \beta$ since $c_{i,\alpha} c_{i,\alpha} = 0$, split n^2-term

line 6: first and last term cancel since $n_{i,\alpha} n_{i,\alpha} = n_{i,\alpha}$ for fermions, second term reorders to be identical to third since $\alpha \neq \beta$

b) First, let us rewrite the action a little

$$\begin{aligned}
S_H[c] &= \int_0^\beta \mathrm{d}\tau \left(\sum_i \sum_\alpha c_{i,\alpha}^\dagger \partial_\tau c_{i,\alpha} - \sum_{\langle i,j\rangle}\sum_\alpha t_{ij} c_{i,\alpha}^\dagger c_{j,\alpha} + U\sum_i n_{i,\uparrow} n_{i,\downarrow} - \mu_0 \sum_i \sum_\alpha n_{i,\alpha}\right) \\
&= \int_0^\beta \mathrm{d}\tau \left(\sum_{ij\alpha\alpha'} c_{i,\alpha}^\dagger \left(\delta_{ij}\delta_{\alpha\alpha'}\left(\partial_\tau - \mu_0\right) - \delta_{\alpha\alpha'} t_{ij}\right) c_{j,\alpha'} + \underbrace{U\sum_i n_{i,\uparrow} n_{i,\downarrow}}_{=:S_U[c]}\right),
\end{aligned}$$

where we defined $t_{ij} = 0$ except for nearest neighbors. Now focusing on the exponentiated on-site interaction term, use the relation from a)

$$\begin{aligned}
\exp\left(-S_U[c]\right) &= \exp\left(\int_0^\beta U\sum_i \left(\frac{2}{3}\mathbf{S}_i^2 - \frac{1}{2}\left(n_{i,\uparrow} + n_{i,\downarrow}\right)\right)\mathrm{d}\tau\right) \\
&= \exp\left(-\int_0^\beta \sum_{ijab} \underbrace{\left(\mathrm{i}S_i^a\right)}_{A^*} \underbrace{\left(\frac{2}{3}U\delta_{ab}\delta_{ij}\right)}_{V} \underbrace{\left(\mathrm{i}S_j^b\right)}_{B} \mathrm{d}\tau\right) \exp\left(-\int_0^\beta \sum_i \frac{U}{2}\left(n_{i,\uparrow} + n_{i,\downarrow}\right)\mathrm{d}\tau\right).
\end{aligned}$$

Now using the Hubbard-Stratonovich transformation

$$\mathrm{e}^{-A^*VB} = \int \mathcal{D}\phi\, \mathrm{e}^{-\phi^* V^{-1}\phi + \mathrm{i}A^*\phi + \mathrm{i}\phi^* B}$$

on the first term (using a three-component real field), and writing the second term in terms of $c_{i,\alpha}$, $c^\dagger_{i,\alpha}$, gives

$$\mathcal{N}\int \mathcal{D}\phi \exp\left(\int_0^\beta -\sum_{ijab}\phi^*_{ai}\frac{3}{2U}\delta_{ab}\delta_{ij}\phi_{bj} - \sum_{ia} S^a_i\phi_{ai} - \sum_{ia} S^a_i\phi^*_{ai}\,\mathrm{d}\tau\right)\exp\left(-\int_0^\beta \sum_{ij\alpha\alpha'} c^\dagger_{i,\alpha}\left(\frac{U}{2}\delta_{ij}\delta_{\alpha\alpha'}\right)c_{j,\alpha'}\,\mathrm{d}\tau\right)$$
$$=\mathcal{N}\int \mathcal{D}\phi \exp\left(\int_0^\beta -\sum_{ia}\phi^2_{ai}\frac{3}{8U} - \sum_{ia} S^a_i\phi_{ai} - \sum_{ij\alpha\alpha'} c^\dagger_{i,\alpha}\left(\frac{U}{2}\delta_{ij}\delta_{\alpha\alpha'}\right)c_{j,\alpha'}\,\mathrm{d}\tau\right),$$

where we did a variable transformation $\phi \to \frac{1}{2}\phi$ and used that ϕ is actually real. Including the other terms of S_H, and using the definition of $\mathbf{S}$ gives:

$$S_H[c,\phi] = \int_0^\beta \mathrm{d}\tau\left(\sum_{ij\alpha\alpha'} c^\dagger_{i,\alpha}\left(\delta_{ij}\delta_{\alpha\alpha'}\left(\partial_\tau \underbrace{-\mu_0 + \frac{U}{2}}_{=:-\mu}\right) - \delta_{\alpha\alpha'}t_{ij} + \delta_{ij}\frac{1}{2}\sum_a \sigma^a_{\alpha\alpha'}\phi_{ai}\right)c_{j,\alpha'} + \frac{3}{8U}\sum_{ia}\phi^2_{ai}\right)$$
$$= \int_0^\beta \mathrm{d}\tau\left(\sum_{ij\alpha\alpha'} c^\dagger_{i,\alpha}\left(\left((\partial_\tau - \mu)\,\delta_{\alpha\alpha'} + \frac{1}{2}\sum_a \phi_{ai}\sigma^a_{\alpha\alpha'}\right)\delta_{ij} - t_{ij}\delta_{\alpha\alpha'}\right)c_{j,\alpha'} + \frac{3}{8U}\sum_{ia}\phi^2_{ai}\right),$$

which is the desired result.

c) Looking again at the exponentiated on-site interaction term, we do the Hubbard-Stratonovich transformation directly with a one-component complex field ϕ without using a)

$$\exp\left(-S_U[c]\right) = \exp\left(-\int_0^\beta U\sum_i n_{i,\uparrow}n_{i,\downarrow}\,\mathrm{d}\tau\right) = \exp\left(-\int_0^\beta \sum_{ij}\underbrace{n_{i,\uparrow}}_{A^*}\underbrace{(\delta_{ij}U)}_{V}\underbrace{n_{j,\downarrow}}_{B}\,\mathrm{d}\tau\right)$$
$$= \mathcal{N}\int \mathcal{D}\left(\phi^*,\phi\right)\exp\left(\int_0^\beta -\sum_{ij}\phi^*_i\left(\delta_{ij}U\right)^{-1}\phi_j + \sum_i\left(\mathrm{i}n_{i,\uparrow}\phi_i + \mathrm{i}\phi^*_i n_{i,\downarrow}\right)\mathrm{d}\tau\right)$$
$$= \mathcal{N}\int \mathcal{D}\left(\phi^*,\phi\right)\exp\left(-\int_0^\beta \sum_i\left(\frac{1}{U}|\phi_i|^2 - \left(\mathrm{i}n_{i,\uparrow}\phi_{i1} - n_{i,\uparrow}\phi_{i2} + \mathrm{i}\phi_{i1}n_{i,\downarrow} + \phi_{i2}n_{i,\downarrow}\right)\right)\mathrm{d}\tau\right)$$
$$= \mathcal{N}\int \mathcal{D}\left(\phi^*,\phi\right)\exp\left(-\int_0^\beta \sum_i\left(\frac{1}{U}|\phi_i|^2 - \left(\mathrm{i}(n_{i,\uparrow} + n_{i,\downarrow})\,\phi_{i1} - (n_{i,\uparrow} - n_{i,\downarrow})\,\phi_{i2}\right)\right)\mathrm{d}\tau\right)$$
$$= \mathcal{N}\int \mathcal{D}\left(\phi^*,\phi\right)\exp\left(-\int_0^\beta \sum_i\left(\frac{1}{U}|\phi_i|^2 - \mathrm{i}n_i\phi_{i1} + 2S^z_i\phi_{i2}\right)\mathrm{d}\tau\right).$$

The real part of the Hubbard-Stratonovich field (ϕ_{i1}) now couples to $n_{i,\uparrow} + n_{i,\downarrow} = n_i$, the occupation number. The imaginary part (ϕ_{i2}), on the other hand, couples to $n_{i,\uparrow} - n_{i,\downarrow} = 2S^z_i$, (twice) the spin in z-direction.