Solution to Exercise 8: Quantum Phase Transition

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Solution to 1(d): Dimensional Analysis

$$
\begin{aligned}
S =& \sum_{\mathbf{k},\omega_n} \phi^*_{\mathbf{k}}(i\omega_n)(r_0 + \mathbf{k}^2 + |\omega_n|^{\frac{2}{z}})\phi_{\mathbf{k}}(i\omega_n) \\
&+ \frac{g}{4\beta\mathcal{V}} \sum_{\omega_{n_1},\omega_{n_2},\omega_{n_3}} \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \phi^*_{\mathbf{k}_1}(i\omega_{n_1})\phi^*_{\mathbf{k}_2}(i\omega_{n_2})\phi_{\mathbf{k}_3}(i\omega_{n_3})\phi_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3}(i\omega_{n_1} + i\omega_{n_2} - i\omega_{n_3})
\end{aligned}
$$

where β is the temperature and $\mathcal{V}$ is the volume.

The dimensional analysis needs to start from the action itself. Each term in the action is dimensionless (we've set $\hbar = 1$). We have the following three conclusions then:

1. $\mathbf{k}$ and $\omega_n^{\frac{1}{z}}$ have the same dimension, and ω_n scales like the temperature T. This means $\mathbf{k}$ and $T^{\frac{1}{z}}$ have the same dimension. We must therefore demand that $[L^{-1}]$ and $[T^{\frac{1}{z}}]$ have the same dimension. (Important: This leads to an anomalous time dimension in general, since T is inverse time, effectively leading to a model in $d+z$ dimensions instead of $d+1$!)
2. The first term has the dimension: $\frac{1}{L^2}[\phi]^2$. In order to be dimensionless, $[\phi]$ must have the dimensions of L or $T^{-\frac{1}{z}}$.
3. The second term has the dimensions of $\frac{[g]T}{L^d}[\phi]^4$. This implies the dimensions of g are same as that of $\frac{L^d}{T}[\phi]^{-4}$ or $T^{\frac{4-(d+z)}{z}}$.

We next analyze the Dyson equation. In the self consistent Hartree-Fock method we have,

$$
r_0 - r = \delta r(r,T) = \Sigma_{SC} = -\frac{g}{\beta\mathcal{V}} \sum_{\mathbf{k},\omega_n} G_{\mathbf{k}}(i\omega_n). \tag{1}
$$

Therefore, $\delta r(r,T)$ has the dimensions of $T^{\frac{2}{z}}$.

Remembering that $\delta r(r,T)$ is a function of two variables, and that for small r and small T, one can write the mass correction as

$$
\delta r(r,T) = \delta r(r,T=0) + \underbrace{f(r,T)g}_{\text{Finite temp correction}} + \ldots,
$$

where $f(r,T)$ is some function of r and T.

Observe that we arranged the mass correction due to interaction in powers of g, but chose to retain only the first order term in g – this is a reflection of the fact that we're doing first order perturbation theory (self-consistently, however). Furthermore, we can set $r = 0$ and $T = T_c$ in $f(r, T)$ since we are interested in criticality. Keeping this in mind we write,

$$\delta r(0, T_c) \approx \delta r(0, T = 0) + \underbrace{a(T_c) g}_{\text{Finite temp correction}} . \tag{2}$$

The dimension of $a(T_c) = f(r = 0, T_c)$ must, therefore, be same as the dimension of $\frac{\delta r(0, T_c)}{g}$, or simply $T^{\frac{(d+z)-2}{z}}$. Just from dimensional analysis, we conclude that

$$a(T_c) = \kappa T_c^{\frac{(d+z)-2}{z}}, \tag{3}$$

for some constant κ.

From previous parts in the tutorial sheet, we also know that $\delta r(0, T = 0) = r_0^c$ at $T = 0$. Hence,

$$\delta r(0, T_c) \approx r_0^c + \kappa T_c^{\frac{(d+z)-2}{z}} g. \tag{4}$$

Feeding this into the criticality condition, $r_0 = \delta r(0, T)$, we obtain,

$$r_0 \approx r_0^c + \kappa T_c^{\frac{(d+z)-2}{z}} g.$$

Thus we conclude,

$$\boxed{T_c \propto |r_0 - r_0^c|^{\frac{z}{d+z-2}}}$$

thereby yielding $\psi = \frac{z}{d+z-2}$.

This matches with the result in Achim's paper on magnetic instabilities of Fermi systems.

In this case, recall that we restricted ourselves to $d + z > 4$ – this is necessary because only then higher order corrections in g will not become dimensionally stronger at low temperatues.

For $z = 1$, one can also do an explicit calculation (I wrote an incorrect result of the Matsubara summation in the tutorial in this case). The correct calculation is as follows:

$$\begin{aligned}
\delta(r, T) &= -\frac{g}{\beta \mathcal{V}} \sum_{\mathbf{k}, \omega_n} \frac{1}{r + \mathbf{k}^2 + |\omega_n|^2} \\
&= -\frac{g}{\mathcal{V}} \sum_{\mathbf{k}} \frac{1 + 2n_B(\sqrt{r + \mathbf{k}^2})}{2\sqrt{r + \mathbf{k}^2}} \\
&= -\frac{g}{(2\pi)^d} \int_{\mathbf{k}} \frac{1 + 2n_B(\sqrt{r + \mathbf{k}^2})}{2\sqrt{r + \mathbf{k}^2}}
\end{aligned}$$

where n_B is the Bosonic distribution function.

Observe the two contributions: the former is independent of temperature (zero-temperature contribution) and the latter goes to zero as $T \to 0$. We want to find the scaling of the latter when $r = 0$. To this end, note

$$\int_{\mathbf{k}} \frac{n_B(\sqrt{\mathbf{k}^2})}{2\sqrt{\mathbf{k}^2}} \propto T^{d-1}$$

in d dimensions, consistent with the result of our dimensional analysis for $z = 1$.