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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 7

Problem 7: Correlations in two dimensions

In this problem set we are going to compute the correlation function for the XY model in two dimensions. This provides us a nice excuse to review some useful calculations for Gaussian distributions.

(a) We first warm up with a simple calculation about one-dimensional gaussian densities. Let $Z(\gamma)$ be defined as

$$Z(\gamma) = \int \mathrm{e}^{-\frac{x^2}{2\sigma^2}+\gamma x}\mathrm{d}x = Z(0)\langle \mathrm{e}^{\gamma x}\rangle_{\gamma=0}. \tag{1}$$

Compute $Z(\gamma)$ and show that every moment of the gaussian distribution $\mathcal{N}(0,\sigma^2)$ can be computed by the taking its derivatives. For this reason, $\langle \mathrm{e}^{\gamma x}\rangle$ is called the *moment generating function.*

(b) We now move to higher dimensions. Let $\vec{x} \in \mathbb{R}^d$, with a multivariate Gaussian distribution defined by $p(x) \propto \mathrm{e}^{-\vec{x}^T\Pi\vec{x}}$ where Π is a positive definite symmetric matrix. We now want to compute the moment generating function in d dimensions, $Z(\vec{\gamma})/Z(0)$, with $\vec{\gamma} \in \mathbb{R}^d$. By using the transformation: $\vec{x} = \vec{y} + \vec{v}$, with a suitable $\vec{v}$, show that

$$Z(\vec{\gamma}) = Z(0)\,\mathrm{e}^{\frac{1}{2}\vec{\gamma}^T\Sigma\vec{\gamma}}, \tag{2}$$

where $\Sigma = \Pi^{-1}$. In the same way as in (a), show that every moment can be calculated by computing the partial derivatives of Eq. (2). In particular, compute $\langle x_i x_j\rangle_{\gamma=0}$

(c) Let us move to even higher dimensions. We study the infinite-dimensional spaces of scalar functions in $\mathbb{R}^2$, where we extend the objects that we already know in finite dimensions.

$$\begin{cases} \vec{x}^T\vec{y} \to \int_{\mathbb{R}^d} \varphi(r)\psi(r) & \text{Scalar product} \\ \Pi\vec{x} \to \int_{\mathbb{R}^d} \Pi(r,r')\varphi(r')\mathrm{d}r' & \text{Matrix-vector product .} \\ \vec{x}^T\Pi\vec{x} \to \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(r)\Pi(r,r')\varphi(r')\mathrm{d}r\,\mathrm{d}r' & \text{Quadratic form} \end{cases} \tag{3}$$

In addition, the role of the identity is played by the Dirac-delta function in virtue of its definition $\varphi(x) = \int_{\mathbb{R}^d} \delta(x-y)\varphi(y)$. We need only a final step in order to compute our correlation functions. If we integrate by parts, we see that the Hamiltonian $\mathcal{H} = \kappa\int_r (\nabla\varphi(r))^2$ can be written as $-\kappa\int_r \varphi(r)\nabla^2\varphi(r)$, which is effectively a quadratic form defined by the linear operator ∇^2. We say that the probability of the field $\varphi(r)$ is given by

$$P[\varphi(r)] \propto \mathrm{e}^{-\kappa\int\varphi\nabla^2\varphi}. \tag{4}$$

In order to repeat the calculation in (b) we need to find its inverse $G(r,r')$, defined by:

$$-\kappa\nabla^2 G(r-r') = \delta(r-r'), \tag{5}$$

where we have expressed $G(r,r') = G(r-r')$ due to the translational invariance of the problem. Use Gauss' Theorem in a sphere of radius R centered in r' to show that $G(r-r')$ in d dimensions decays as $R^{-(d-2)}$ and that in the case of $d=2$, $G(R) = -\frac{1}{2\pi\kappa}\log\left(\frac{R}{a}\right) + \text{const.}$

(d) All the ingredients necessary to compute the spin-spin correlation functions are now ready. Compute the value of $\langle e^{i(\varphi(r_1)-\varphi(r_2))}\rangle$ and from there deduce the spin-spin correlation $\langle S(r_1)\cdot S(r_2)\rangle$. *Hint: You may want to use the relation* $\varphi(r_i) = \int_r \delta(r-r_i)\varphi(r)$, *considering this as a scalar product.*

To be discussed on: Mon, December 16

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`