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New hierarchies of multicriticality in two-dimensional field theory

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The minimal conformal model $M_{p,q}$, perturbed by the relevant scaling field $\phi_{1,3}$, is argued to undergo a crossover to the model $M_{p-(q-p),q-(q-p)}$, at least for large values of $p/(q-p)$. Hence its critical manifold is nested into all manifolds of lower criticality.

The principle of conformal invariance has yielded a partial classification of universal critical behavior in two dimensions: the "minimal" conformal theories $M_{p,q}$ (where p and q are to relatively prime positive integers with $p < q$) exhaust all universality classes with a finite number of primary scaling fields [1-3]. These theories have central charge $c = 1-6(q-p)^2/qp$. All scaling fields belong to degenerate representations of the Virasoro algebra; their correlation functions are bilinear forms of generalized hypergeometric functions [1,4]. These forms are determined by modular invariance of the partition function on the torus [2,5]; we restrict ourselves here to "diagonal" theories, which have the primary fields $\phi_{r,s}$ with $1 \leqslant r \leqslant p-1$ and $1 \leqslant s \leqslant q-1$.

The physical interpretation of these universality classes is tied to the understanding of their scaling behavior close to criticality and of the global topology of the embedding phase diagram.

For the subset of *unitary* minimal theories $M_p \equiv M_{p,p+1}$ ($p = 3,4,\ldots$) [6,7], that interpretation is clear by now: the theory M_p describes the Landau-Ginzburg model of a single scalar field with the lagrangian [8]

$$\mathcal{L} = (\nabla\Phi)^2 + \sum \lambda_j \Phi^j \tag{1}$$

at the $(p-1)$-critical point $\lambda_1 = \lambda_2 = \ldots = \lambda_{2m-3} = 0$, $\lambda_{2m-2} \neq 0$. As one would expect from this picture, the manifold $\mathcal{C}_p$ of $(p-1)$-criticality is nested into all manifolds of lower criticality [9,3]:

$$\ldots \subset \mathcal{C}_p \subset \mathcal{C}_{p-1} \subset \ldots \subset \mathcal{C}_3. \tag{2}$$

This infinite hierarchy of multicriticality appears in the Landau-Ginzburg model already at the mean field level. The strong fluctuations in two dimensions, while drastically modifying the critical exponents, do not alter the topology of the phase diagram.

In statistical mechanics, however, unitarity is not as crucial a requirement as it is in quantum field theory. Several physically interesting models are manifestly *nonunitary*, for example the O(N)-model in the limit $N \to 0$ (polymers in solution) [10,4], the Q-state Potts model for $Q \to 1$ (percolation) [11,4], and the N-replica Q-state Potts model for $N \to 0$ (quenched bond disorder) [3]. These theories are not minimal; their operator algebra contains infinitely many (degenerate and nondegenerate) primary fields.

Some nonunitary minimal models attracted considerable attention as well, since the simplest multicritical matrix models [12] represent these theories coupled to two-dimensional quantum gravity [13].

The simplest nonunitary minimal model is the Yang-Lee theory $M_{2,5}$ [14]. It describes the critical point of an Ising model in a purely imaginary magnetic field and corresponds to a Landau-Ginzburg lagrangian of the form $\mathcal{L} = (\nabla\Phi)^2 + i\lambda_3\Phi^3$ [15].

The only relevant perturbation of the Yang-Lee model leads to a massive phase whose scaling region is characterized by a purely elastic scattering theory [16]. Some perturbations of higher nonunitary minimal models are known to generate well-defined massive phases as well [17,18], but this does not say which of these universality classes are connected by massless

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crossover phenomena and hence are part of a common phase diagram. In this letter, we argue that there are infinitely many hierarchies of nested multicriticality

$$\ldots \subset \mathcal{C}_{p,q} \subset \mathcal{C}_{p-n,q-n} \subset \mathcal{C}_{p-2n,q-2n} \subset \ldots, \tag{3}$$

where $n \equiv q - p$. Each crossover between two subsequent theories in a sequence (3) reduces the number $m \equiv p/(q-p)$ by one. Since the central charge, the scaling dimensions and the operator product coefficients of $M_{p,q}$ depend analytically on m alone, we call m the "analytic index" of $M_{p,q}$. The number n is common to all theories in a sequence and is called the "nonunitarity index" for reasons that will become apparent below.

Recall the argument based on perturbation theory in the parameter $1/m$ that leads to the hierarchy (2) for the unitary series [9,3]. The field $\phi_{1,3}$, which is the least relevant scaling field in a unitary minimal model, enjoys the following properties: (a) it has scaling dimension $x_{1,3} = 2 - 4/m + O(1/m^2)$ and becomes marginal as $m \to \infty$; (b) it has a finite operator product coefficient with itself [19],

$$C(m) = \frac{4}{\sqrt{3}} + O\left(\frac{1}{m}\right), \tag{4}$$

but does not couple to any other relevant field. Hence the perturbation of M_p by this field has the beta function

$$\beta(u) = -\frac{4}{m}u + \frac{4}{\sqrt{3}}u^2 + O\left(\frac{1}{m^3}\right). \tag{5}$$

The infrared fixed point $u^* = \sqrt{3}/m + O(1/m^2)$ describes a critical theory with central charge

$$c^* = c_m - \frac{12}{m^3} + O\left(\frac{1}{m^4}\right) = c_{m-1} + O\left(\frac{1}{m^4}\right). \tag{6}$$

Any unitary theory with $c < 1$ has to be minimal [6]; since the perturbation preserves unitarity and $c^* < c_m < 1$ [#1], the infrared theory has to be a member of the same series. This must be M_{p-1}, which is the only unitary theory whose central charge satisfies eq. (6). The perturbative calculation has been confirmed by supersymmetry arguments for the crossover from the tricritical to the critical Ising model (the case $p =$ 4) [21] and more recently by thermodynamic Bethe ansatz methods for general values of p [22].

[#1] The c-theorem [20,9] ensures that this relation is valid beyond perturbation theory.

We wish to show that in any minimal model $M_{p,q}$ with $m \gg 1$, the field $\phi_{1,3}$ induces a crossover to the model $M_{p-n,q-n}$. Since the dimension $x_{1,3}$ and the operator product coefficient (4) depend analytically on m alone, the perturbative beta function is the same as in the unitary case, and eq. (6) is still valid. This is consistent with the assertion but does not prove it since (a) it is not clear a priori whether the infrared theory is minimal and (b) even the minimal theories fill the interval $-\infty < c < 1$ densely.

To analyze the field content of the perturbed theory, it is convenient to label the primary fields $\phi_{r,s}$ of $M_{p,q}$ in order of decreasing relevance by the index $N \equiv qr - ps$, which measures the distance of the point (r,s) to the diagonal of the Kac table and takes the values $N = 1,2,\ldots,p-1,p+1,\ldots,q-1,q+1,\ldots,2p-1,2p+1,\ldots,p+q-1,p+q+1,\ldots$ (the omitted values correspond to points outside the Kac table). The field ϕ_N has scaling dimension

$$\begin{aligned} x_N(m) &= \frac{N^2 - n^2}{2n^2 m(m+1)} \\ &= \frac{N^2 - n^2}{2n^2 m^2}\left[1 + O\left(\frac{1}{m}\right)\right]. \end{aligned} \tag{7}$$

Hence all $p+q-4$ fields ϕ_N with $1 \leqslant N \leqslant p+q-1$ are relevant. The first $n-1$ of them have negative scaling dimension; in this sense, n measures the degree of nonunitarity of $M_{p,q}$.

The operator product coefficients $C_N \equiv C_{NN(1,3)}$ are [19]

$$\begin{aligned} C_N(m) &= -\frac{\Gamma((N+p)/q)\Gamma((-N+p)/q)}{\Gamma((-N+q-p)/q)\Gamma((N+q-p)/q)} \\ &\quad \times \left[\frac{1}{2\sqrt{3}} + O\left(\frac{1}{m}\right)\right] \\ &= x_N(m)\left[\frac{1}{\sqrt{3}} + O\left(\frac{1}{m}\right)\right]; \end{aligned} \tag{8}$$

the last equality is valid for generic values of N. For $N \approx kq$ with $k = 1,2,\ldots$, $C_N(m)$ is enhanced by the finite factor $(N-kq-n)/(N-kq+n)$; these are just the fields that mix appreciably under the renormalization group [9]. For generic values of N, the mixing is of $O(1/m)$, and eqns. (4) and (8) determine directly the infrared scaling dimension of ϕ_N,

$$x_N{}^* = x_N(m) + \frac{2C_N(m)}{C(m)}\left[\frac{4}{m} + O\left(\frac{1}{m^2}\right)\right]$$
$$= x_N(m)\left[1 + \frac{2}{m} + O\left(\frac{1}{m^2}\right)\right]$$
$$= x_N(m-1) + O\left(\frac{1}{m^2}\right). \qquad (9)$$

In particular, the infrared theory has exactly $n-1$ scaling fields of negative dimension: the crossover conserves the nonunitarity index n. This, together with eq. (6), proves the assertion provided the infrared theory is minimal at all. The above considerations give evidence for this as well. Since the operator product coefficients (4) and (8) are real, the infrared scaling dimensions (9) are still real. This remains true for fields that mix under the renormalization group (whose infrared dimensions are obtained as the eigenvalues of real symmetric matrices) and for descendant fields (whose operator product coefficients are given in terms of the coefficients their primaries and real polynomials in the scaling dimensions and the central charge). Hence the partition function on the torus of the infrared theory has positive Boltzmann weights. But a non-minimal theory with $c < 1$ must have negative Boltzmann weights, as follows from modular invariance [2,3].

Some scaling fields of the infrared theory can be shown explicitly to be degenerate Virasoro primaries. For example, the field $\phi_{1,2}$ of $M_{p,q}$ has dimension $x_{1,2}(m) = \frac{1}{2}(1 - 3/m) + O(1/m^2)$ and belongs to the subalgebra of fields $\phi_{1,s}$, which is preserved under the perturbation by $\phi_{1,3}$. To leading order, it does not mix with any other field under the renormalization group and maps, according to eq. (9), onto an infrared primary field ϕ^* of dimension $x^* = \frac{1}{2}(1 + 3/m) + O(1/m^2) = x_{2,1}(m-1) + O(1/m^2)$. Its only left descendant at level two, $L_{-1}^2\phi_{1,2}$, maps onto an infrared field of dimension $x^* + 2 + O(1/m^2)$, and that is the only field of the infrared subalgebra with that dimension and angular momentum 2. Therefore ϕ^* must be degenerate at level two, as expected for the field $\phi_{2,1}$ of $M_{p-n,q-n}$.

Clearly, the main result (3) is not rigorously founded since it is based on first order perturbation theory. In particular for small values of m, variations in the flow pattern are to be expected. Therefore our arguments deserve to be checked by nonperturbative methods. A few additional comments are in order.

Any minimal model $M_{p,q}$ with $p > 2$ enjoys a Z_2 symmetry

$$\phi_{r,s} \to (-)^{r-1}\phi_{r,s} \quad \text{if } p \text{ is even,}$$
$$\phi_{r,s} \to (-)^{s-1}\phi_{r,s} \quad \text{if } q \text{ is even,}$$
$$\phi_{r,s} \to (-)^{s-r}\phi_{r,s} \quad \text{if } q-p \text{ is even,} \qquad (10)$$

that leaves its operator algebra invariant. Since $\phi_{1,3}$ is always even, the crossover to $M_{p-n,q-n}$ preserves this symmetry, which is consistent with the flow of fields given by eq. (9).

Exactly $2n$ primary fields become irrelevant under the crossover. These are the $2n-1$ fields $\phi_{2p-n} \equiv \phi_{1,3}, \phi_{2p-n+1}, \ldots, \phi_{2p-1}, \phi_{2p+1}, \ldots, \phi_{p+q-1}$, which have scaling dimensions $x_{1,3} \leqslant x_N < 2$, and the field $\phi_{2p-2n} \equiv \phi_{2,4}$, which leaves the Kac table as it does in the unitary case.

We have limited ourselves to diagonal minimal theories. It would be very interesting to disentangle the flows between the other modular invariants, which has recently been achieved for the unitary series [23].

Under the crossovers considered here, which preserve the nonunitarity index n, the central charge decreases. This does not rule out crossovers between different sequences (3) that may violate the c-theorem. In fact, the mean-field analysis suggests that the usual Ising fixed point is linked to the Yang–Lee fixed point through a crossover induced by the imaginary magnetic field and the reduced temperature in a fine-tuned linear combination. More generally, we should expect nonunitary minimal universality classes to appear in the phase diagram of the Landau–Ginzburg model (1) when some thermodynamic parameters λ_j take imaginary values. All that hints at an extremely rich scenario yet to be discovered. Hopefully, this will help to answer the question whether some of these universality classes occur in nature.

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