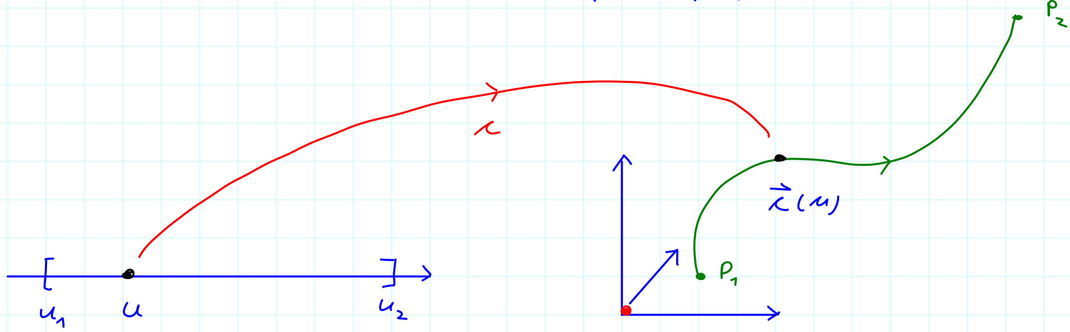
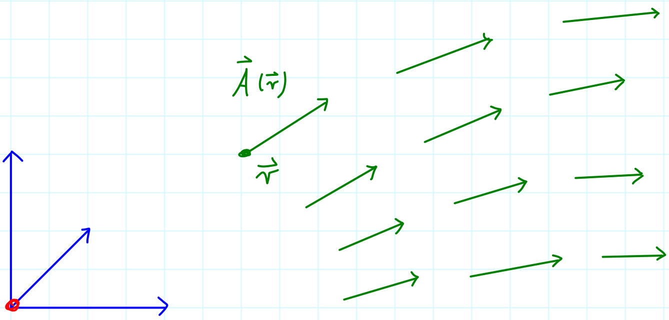
<u>Letzte Vrlsg.:</u>

- <u>Parametrisierung</u> eines <u>Wegs</u> / einer <u>Kurve</u> im $\mathbb{R}^3$

$$\hat{=} \quad \text{Abb.} \quad r : [u_1, u_2] \to \mathbb{R}^3, \quad u \mapsto \vec{r}(u)$$



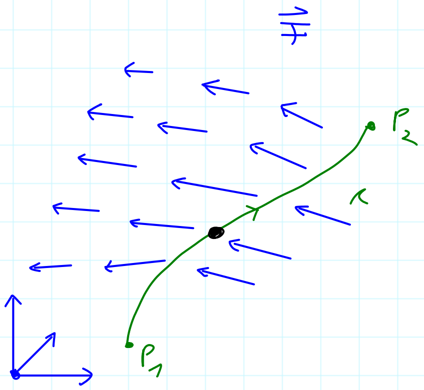
- <u>Vektorfeld</u> $\hat{=}$ Abb. $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{r} \mapsto \vec{A}(\vec{r})$



- <u>Wegintegral/Kurvenintegral</u> des Vektorfelds $\vec{A}$ längs Weg r:

$$\boxed{\int_r \vec{A}\, d\vec{l} := \int_{u_1}^{u_2} \langle \vec{A}(\vec{r}(u)), \vec{r}\,'(u) \rangle \, du}$$

(Parametr. $r : [u_1, u_2] \to \mathbb{R}^3$, $u \mapsto \vec{r}(u)$)

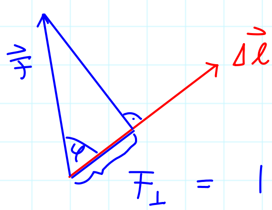


Motivation:

benötigte Arbeit W für Transport des Teilchens von P_1 nach P_2 längs Weg r unter Wirkung der Kraft $\vec{F}(\vec{r})$:

$$W = \sum_{i=1}^{N} \Delta W_i = \sum_{i=1}^{N} -\langle \vec{F}_i, \vec{\Delta l}_i \rangle$$

$$\underset{\|\Delta \vec{l}_i\| \to 0}{=} -\int_{u_1}^{u_2} \langle \vec{F}(\vec{r}(u)), \vec{r}\,'(u) \rangle \, du \overset{\text{Def.}}{=} -\int_{r} \vec{F} \, d\vec{l}$$



$$F_\perp = |\vec{F}| \cos\varphi$$

$$\rightarrow \quad \Delta W \underset{\text{Def.}}{=} -F_\perp \cdot |\vec{\Delta l}| = -|\vec{F}|\,|\vec{\Delta l}| \cos\varphi = -\langle \vec{F}, \vec{\Delta l} \rangle$$

Partielle Ableitungen, Gradient, Differenzial

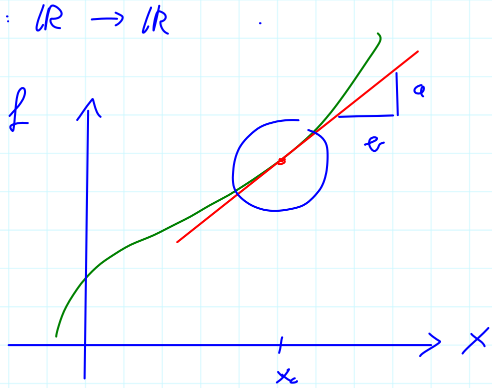
⇒ Potenzial eines Vektorfeldes, konservatives Vf.

≃ Ableitung in d = 1 !

$d=1$ Ableitung einer Fkt. $f: \mathbb{R} \to \mathbb{R}$.

$$f'(x_0) = \frac{a}{b} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\underset{h=\Delta x}{=} \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$



⇒ lineare Näherung (von f in x_0):

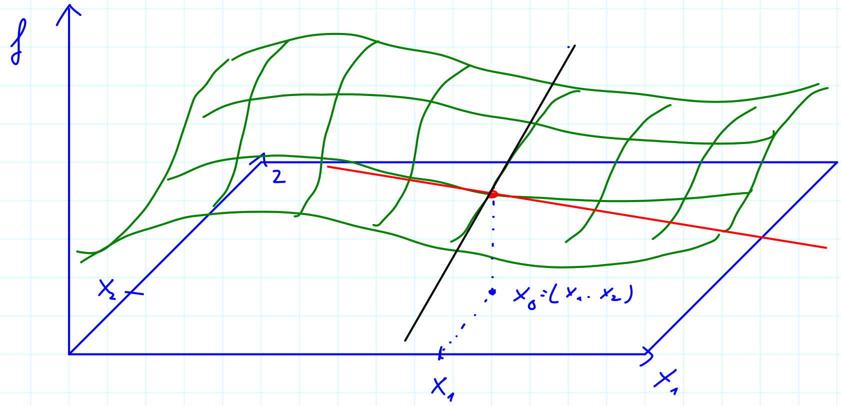
$$f(x_0+h) = f(x_0) + f'(x_0)\,h + \mathcal{O}(h^2)$$

Verallgemeinerung für Fkt. f in n Variablen:

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{x} = (x_1, x_2, \dots x_n) \longmapsto f(x_1, x_2, \dots, x_n) = f(\vec{x})$$

$n=2$:



Steigung der Tangente von f in $\vec{x}_0$ in x_1 - Richtung
x_2 - " "

= partielle Ableitung von f in x_0 nach x_1
x_2

$$\frac{\partial f}{\partial x_1}(\vec{x}_0) \qquad \left(f'(x_0) = \frac{d f}{d x}(x_0) \right)$$

$$\frac{\partial f}{\partial x_2}(\vec{x}_0) \qquad \text{"}\partial\text{" = "del" = "delta")}$$

$\underline{\underline{n=2}}$ $\qquad f(x_1, x_2)$;

$$\frac{\partial f}{\partial x_1}(\vec{x}_0) := \Big(f(x, x_2) \Big)' = \lim_{h \to 0} \frac{1}{h}\big(f(x+h, x_2) - f(x_1, x_2)\big)$$

$$\frac{\partial f}{\partial x_2}(\vec{x}_0) := \Big(f(x_1, x) \Big)' = \text{" \quad " \quad "}$$

Def.: Partielle Ableitungen von $f: \mathbb{R}^n \to \mathbb{R}$ in $\vec{x}_0$:

$$\frac{\partial f}{\partial x_i}(\vec{x}_0) := \Big(f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \Big)'(x = x_i)$$

$$= \lim_{h \to 0} \frac{1}{h}\big(f(\dots, x_i + h, \dots) - f(\dots, x_i, \dots) \big)$$

$$= \lim_{h \to 0} \frac{1}{h}\big(f(\vec{x}_0 + h\vec{e}_i) - f(\vec{x}_0) \big)$$

Bsp.: $\quad f(x_1, x_2, x_3) = 2x_1^2 (3x_2 + \cos x_3)$

$$\frac{\partial f}{\partial x_1}(\vec{x}) = 4x_1 (3x_2 + \cos x_3)$$

$$\frac{\partial f}{\partial x_2}(\vec{x}) = 2x_1^2\, 3 = 6x_1^2$$

$$\frac{\partial f}{\partial x_3}(\vec{x}) = -2x_1^2 \sin x_3$$

→ lineare Näherung der Fkt. $f : \mathbb{R}^n \to \mathbb{R}$ in $\vec{x}_0$:

$$f(\vec{x}_0 + \vec{h}) = f(x_1 + h_1, x_2 + h_2, \dots, x_n + h_n)$$

$\|\vec{h}\| \ll 1$

$$= f(x_1, x_2 + h_2, \dots, x_n + h_n) + \frac{\partial f}{\partial x_1}(\vec{x}_0)\, h_1 + \sigma(\|\vec{h}\|^2)$$

$$\vec{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$= f(x_1, x_2, x_3 + h_3, \dots) + \frac{\partial f}{\partial x_1}(\vec{x}_0)\, h_1 + \frac{\partial f}{\partial x_2}(\vec{x}_0)\, h_2 + \sigma(\|\vec{h}\|^2)$$

$$\to \quad f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \underbrace{\frac{\partial f}{\partial x_1}(\vec{x}_0)\, h_1 + \dots + \frac{\partial f}{\partial x_n}(\vec{x}_0)\, h_n} + \sigma(\|\vec{h}\|^2)$$

$$= f(\vec{x}_0) + \underbrace{\sum_{i=1}^{n} \boxed{\underbrace{\frac{\partial f}{\partial x_i}(\vec{x}_0)}_{a_i}}\; \underbrace{h_i}_{b_i}}_{\sum_{i=1}^{n} a_i b_i \;=\; \langle \vec{a}, \vec{b} \rangle} + \sigma(\|\vec{h}\|^2)$$

mit $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

→ **Def.** Gradient der Fkt. $f : \mathbb{R}^n \to \mathbb{R}$ in $\vec{x}_0$:

$$\operatorname{grad} f(\vec{x}_0) := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}_0) \cdot \vec{e}_i = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) \\ \frac{\partial f}{\partial x_2}(\vec{x}_0) \\ \vdots \end{pmatrix}$$

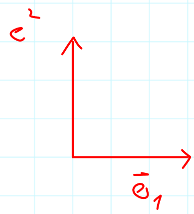
$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, ...

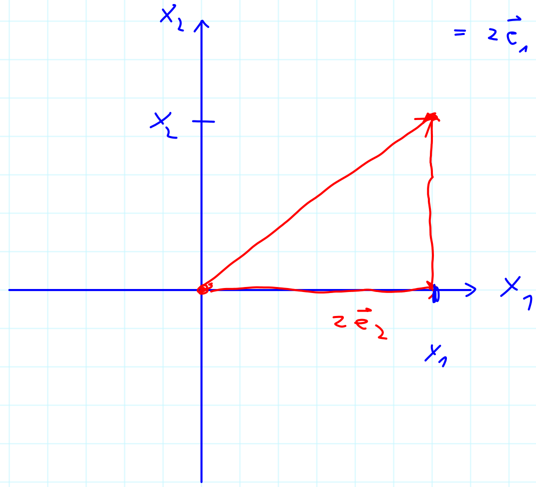
2) lineare Näherung von f in $\vec{x}_0$:

$$\boxed{f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \langle \operatorname{grad} f(\vec{x}_0), \vec{h} \rangle + \sigma(h^2)}$$

$\ulcorner$ $\mathbb{R}^2$

$$\vec{x} = \begin{pmatrix} 2 \\ 1{,}5 \end{pmatrix} = 2\vec{e}_1 + 1{,}5\,\vec{e}_2$$





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Bsp: $\quad f(x_1, x_2, x_3) = (x_1 + 1)^2 (2x_2 + \cos x_3)$

$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$: lin. Näherung von f in $\vec{0}$:

$$\rightsquigarrow \quad \operatorname{grad} f(\vec{0}) = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \quad ; \quad f(\vec{0}) = 1$$

$$\frac{\partial f}{\partial x_1}(\vec{0}) = 2(x_1+1)(2x_2 + \cos x_3)\Big|_{\vec{x}=\vec{0}} = 2$$

$$\frac{\partial f}{\partial x_2}(\vec{0}) = (x_1+1)^2 \cdot 2 \Big|_{\vec{x}=\vec{0}} = 2$$

$$\frac{\partial f}{\partial x_3}(\vec{0}) = (x_1 + 1)^2(-\sin x_3)\Big|_{\vec{x}=\vec{0}} = 0$$

$$\rightsquigarrow \quad f(\vec{0} + \vec{h}) = 1 + \langle \operatorname{grad} f(\vec{0}), \vec{h} \rangle + \mathcal{O}(h^2)$$

$$= 1 + 2h_1 + 2h_2$$

<u>Notationen:</u>

Partielle Ableitungen:

$$\frac{\partial f}{\partial x_i} \equiv \partial_{x_i} f \equiv \partial_i f$$

Gradient: Nabla-Operator:

$$\vec{\nabla} := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \operatorname{grad} f$$

Geometrische Bedeutung des Gradienten

$f: \mathbb{R}^n \to \mathbb{R}$

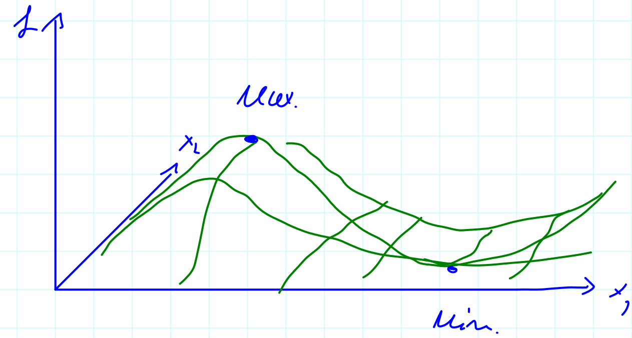
1) $\operatorname{grad} f(\vec{x}_0)$ parallel zur Richtung $\hat{n}_0$ des stärksten Anstiegs von f in $\vec{x}_0$

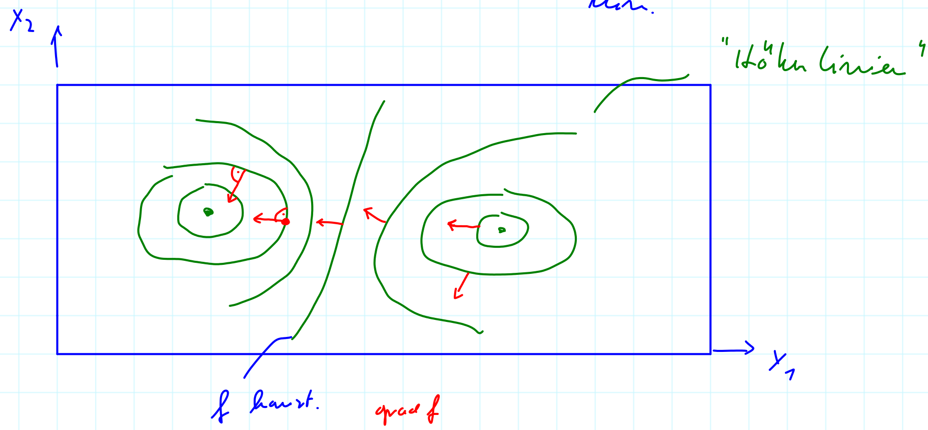
2) $\| \operatorname{grad} f(\vec{x}_0) \|$ Steigung von f in Richtung $\hat{n}_0$

3) $\operatorname{grad} f(\vec{x}_0)$ ist senkrecht zu Niveau-Fläche von f in $\vec{x}_0$

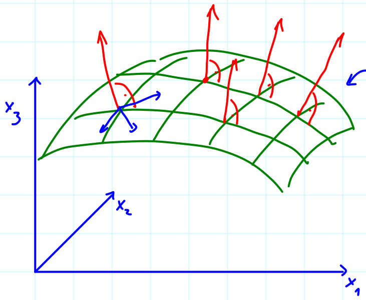
$n = 2$:

f:





$\underline{\underline{n=3}}$



Niveaufläche

$$F_a = \{ \vec{x} \in \mathbb{R}^3 \mid f(\vec{x}) = a \}$$