Lösungshinweise Blatt 11

57) Wir führen die Existenz von $a \in V$ und $b \in W$ mit $a \otimes b = u$ auf einen Widerspruch.

$$a = a_1 v_1 + a_2 v_2 \,, \qquad b = b_1 w_1 + b_2 w_2$$

$$\rightarrow \quad a \otimes b = a_1 b_1\, v_1 \otimes w_1 + a_1 b_2\, v_1 \otimes w_2 + a_2 b_1\, v_2 \otimes w_1 + a_2 b_2\, v_2 \otimes w_2 \overset{!}{=} v_1 \otimes w_1 + v_2 \otimes w_2$$

$\rightarrow$ (i) $a_1 b_1 = 1$

$\wedge$ (ii) $a_1 b_2 = 0$

$\wedge$ (iii) $a_2 b_1 = 0 \rightarrow \underbrace{a_2 = 0}_{\substack{\text{Widerspruch} \\ \text{zu (iv)}}}$ oder $\underbrace{b_1 = 0}_{\substack{\text{Widerspruch} \\ \text{zu (i)}}}$

(iv) $a_2 b_2 = 1$

58) $T = 2\pi \rightarrow \omega_n = \frac{2\pi}{T} n = n$

$$\rightarrow \quad f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \frac{1}{2\pi} \int_0^{\pi} e^{-int}\, dt \,, \quad n \in \mathbb{Z}$$

$n = 0$: $\quad f_0 = \frac{1}{2\pi} \int_0^{\pi} dt = 1/2$

$n \neq 0$: $\quad f_n = \frac{1}{2\pi} \left. \frac{e^{-int}}{-in} \right|_0^{\pi} = \frac{1}{2\pi i n} \left(1 - e^{i\pi n} \right) = \frac{1}{2\pi i n} \left(1 - (-1)^n \right)$

d.h. $\quad f_n = \begin{cases} 0 & : n \text{ gerade} \\ \frac{1}{i\pi} \frac{1}{n} & : n \text{ ungerade} \end{cases}$

$\rightarrow$ Fourierreihe $\quad f(t) = \frac{1}{2} + \sum_{\ell=0}^{\infty} \frac{1}{i\pi} \frac{1}{2\ell+1} \underbrace{\left(e^{i(2\ell+1)t} - e^{-i(2\ell+1)t} \right)}_{= 2i \sin((2\ell+1)t)}$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{\ell=0}^{\infty} \frac{\sin((2\ell+1)t)}{2\ell+1} \,.$$

59)

f: Periode 1 $\rightarrow$ $k_n = 2\pi \cdot n$

$$\rightarrow \quad f_n = \int_0^1 \sum_{m\in\mathbb{Z}} \delta(x-m-d)\, e^{-2\pi i n x}\, dx = e^{-2\pi i n d}$$

$\rightarrow$ Fourierreihe $\quad f(x) = \sum\limits_{n\in\mathbb{Z}} e^{2\pi i n(x-d)}$

g: Periode a, $k_n = \frac{2\pi}{a} n$

$$\rightarrow \quad g_n = \frac{1}{a}\int_{-a/2}^{a/2} \sum_{m\in\mathbb{Z}} \delta(x-am)\, e^{-i k_n x}\, dx = \frac{1}{a}$$

$\rightarrow$ Fourierreihe $\quad g(x) = \frac{1}{a}\sum\limits_{n\in\mathbb{Z}} e^{i\frac{2\pi}{a}n x}$

60)

a) $f = \sum\limits_{n\in\mathbb{Z}} f_n e_n \quad$ wobei $e_n(t) = e^{i\omega_n t}$

mit $\langle e_n, e_m\rangle = \delta_{nm}$ folgt

$$|f|^2 = \langle f, f\rangle = \sum_{n,m\in\mathbb{Z}} f_n^* f_m \langle e_n, e_m\rangle = \sum_{n\in\mathbb{Z}} |f_n|^2 \quad \square$$

b) 2π-per. Fkt. $f(t)$ mit $f(t) = t$ für $t \in\,]-\pi,\pi]$

besitzt nach Vrlsg. Fourierkoeff. en

$$f_n = \begin{cases} 0 & : n = 0 \\ i\,\frac{(-1)^n}{n} & : n \neq 0 \end{cases}$$

$$\rightarrow \quad \sum_{n\in\mathbb{Z}} |f_n|^2 = 2\sum_{n=1}^{\infty}\frac{1}{n^2} \overset{a)}{=} |f|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt =$$

$$= \frac{1}{\pi}\int_0^{\pi} t^2\, dt = \frac{\pi^2}{3}\, ; \quad \text{d.h.} \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad \square$$

61)

$$g(t) = f(t-d) = \sum_{n\in\mathbb{Z}} \hat{f}_n \, e^{i\omega_n (t-d)}$$

$$= \sum_{n\in\mathbb{Z}} \underbrace{e^{-i\omega_n d}\hat{f}_n}_{\overset{!}{=}\hat{g}_n} \, e^{i\omega_n t}$$

$$f(t) = \sum_{n\in\mathbb{Z}} f_n \, e^{i\omega_n t}$$

$$\rightarrow \; h(t) = f'(t) = \sum_{n\in\mathbb{Z}} \underbrace{f_n \, i\omega_n}_{\overset{!}{=}\hat{h}_n} \, e^{i\omega_n t}$$

$$j_n = \frac{s}{T}\int_{-\frac{T}{s}/2}^{+\frac{T}{s}/2} f(st)\, e^{-i\frac{2\pi}{T}snt}\, dt$$

$$\underset{t=\tau/s}{=} \frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, e^{-i\frac{2\pi}{T}\cancel{s}\,n\frac{\tau}{\cancel{s}}}\, d\tau = f_n$$