Lösungshinweise Blatt 13

63) $$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ax^2 - ikx} \underset{\uparrow \atop \text{Gauß}}{=} \frac{1}{\sqrt{2a}}\, e^{-k^2/4a}$$

$$\hat{g}(k) = \int_{0}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-(a+ik)x} = \frac{1}{\sqrt{2\pi}}\, \frac{1}{a+ik}$$

$$\hat{h}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-a|x| - ikx} = \int_{0}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-(a+ik)x} + \int_{0}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-(a-ik)x}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a+ik} + \frac{1}{a-ik}\right) = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+k^2}$$

$$\Rightarrow \quad h(x) = e^{-a|x|} \overset{!}{=} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{h}(k)\, e^{ikx} = \int_{-\infty}^{\infty} dk\, \frac{1}{\pi}\, \frac{a}{a^2+k^2}\, e^{ikx}$$

$$\rightarrow \quad \hat{j}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \frac{1}{\pi}\, \frac{a}{a^2+x^2}\, e^{-ikx} = \frac{1}{\sqrt{2\pi}}\, e^{-a|k|}$$

64) z.B. mittels Methode aus Vrlsg.:

(i)
$$T_0(x) = \vartheta \cos(k_0 x) \xrightarrow{\mathcal{F}} \hat{T}_0(k) = \frac{\vartheta}{2}\sqrt{2\pi}\,\big(\delta(k-k_0) + \delta(k+k_0)\big)$$

$$\xrightarrow{\cdot e^{-ak^2t}} \quad \hat{T}(k,t) = \frac{\vartheta}{2}\sqrt{2\pi}\, e^{-ak^2t}\big(\delta(k-k_0) + \delta(k+k_0)\big)$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad T(x,t) = \vartheta\, e^{-ak_0^2 t} \cos(k_0 x)$$

(ii)
$$T_0(x) = \vartheta\, e^{-x^2/2\sigma_0^2} \xrightarrow{\mathcal{F}} \hat{T}_0(k) = \vartheta\, \sigma_0\, e^{-\frac{k^2\sigma_0^2}{2}}$$

$$\xrightarrow{\cdot e^{-ak^2t}} \quad \hat{T}(k,t) = \vartheta\, \sigma_0\, e^{-k^2\left(\frac{\sigma_0^2}{2} + at\right)}$$

$$\xrightarrow{\mathcal{F}^{-1}} \quad T(x,t) = \vartheta\, \sigma_0 \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{-\left(\frac{\sigma_0^2}{2} + at\right)k^2 + ikx}$$

$$\underset{\uparrow \atop \text{Gauß}}{=} \frac{\vartheta\, \sigma_0}{\sqrt{\sigma_0^2 + 2at}}\, e^{-x^2/2\underbrace{(\sigma_0^2 + 2at)}_{=(\sigma(t))^2}}$$

d.h. $\sigma(t) = \sqrt{\sigma_0^2 + 2at}$.

65)

<u>Gauß-Verteilung:</u> mit $\hat{p}_\sigma(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2\sigma^2}{2}}$ (Vorlesg.)

und Faltungssatz folgt

$$(p_{\sigma_1} * p_{\sigma_2})(x) = \mathcal{F}^{-1}\left[k \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2\sigma_1^2}{2}} \cdot e^{-\frac{k^2\sigma_2^2}{2}} \right](x)$$

$$= \mathcal{F}^{-1}\left[k \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}(\sigma_1^2+\sigma_2^2)} \right](x) = p_{\sigma_1^2+\sigma_2^2}(x) .$$

<u>Cauchy-Verteilung:</u>

nach Aufgabe 63) ist $\hat{q}_a(k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-a|k|}$;

mit Faltungssatz folgt

$$q_{a_1} * q_{a_2}(x) = \mathcal{F}^{-1}\left[k \mapsto \frac{1}{\sqrt{2\pi}} e^{-(a_1+a_2)|k|} \right](x) = q_{a_1+a_2}(x) .$$