Lösungshinweise Blatt 2

6a)

$$\langle e_i, u\rangle = \langle e_i, \sum_{j=1}^{n} u_j e_j\rangle = \sum_{j=1}^{n} u_j \overbrace{\langle e_i, e_j\rangle}^{=\delta_{ij}} = u_i$$

6b)

$$\langle u, v\rangle = \langle \sum_i u_i e_i, \sum_j v_j e_j\rangle = \sum_{ij} u_i^* v_j \overbrace{\langle e_i, e_j\rangle}^{=\delta_{ij}}$$

$$= \sum_i u_i^* v_i$$

7a)

$$|b_1|^2 = \langle b_1, b_1\rangle = \frac{1}{2}(1+1) = 1$$

$$|b_2|^2 = \langle b_2, b_2\rangle = 1$$

$$\langle b_1, b_2\rangle = \frac{1}{2}(1 - i^* i) = 0$$

7b)

$$\langle b_1 + i b_2, b_1 - i b_2\rangle = |b_1|^2 - i^* i\, |b_2|^2 = 0$$

$$\langle i b_1, b_1\rangle + \langle b_2, i b_2\rangle = -i + i = 0$$

$$\langle e^{i\varphi} b_1, e^{i\varphi} b_1\rangle = e^{-i\varphi} e^{i\varphi} = 1$$

$$\langle b_2, e^{i\varphi} b_2\rangle - \langle e^{i\varphi} b_1, b_1\rangle = e^{i\varphi} - e^{-i\varphi} = 2i \sin\varphi$$

7c)

$$u_1 = \langle b_1, u\rangle = \frac{1}{\sqrt{2}}\langle \begin{pmatrix}1\\ i\end{pmatrix}, \begin{pmatrix}i\\ 2\end{pmatrix}\rangle = -i/\sqrt{2}$$

$$u_2 = \langle b_2, u\rangle = \frac{1}{\sqrt{2}}\langle \begin{pmatrix}1\\ -i\end{pmatrix}, \begin{pmatrix}i\\ 2\end{pmatrix}\rangle = 3i/\sqrt{2}$$

$$v_1 = \langle b_1, v\rangle = \frac{1}{\sqrt{2}}\langle \begin{pmatrix}1\\ i\end{pmatrix}, \begin{pmatrix}2+i\\ 1\end{pmatrix}\rangle = \sqrt{2}$$

$$v_2 = \langle b_2, v\rangle = \frac{1}{\sqrt{2}}\langle \begin{pmatrix}1\\ -i\end{pmatrix}, \begin{pmatrix}2+i\\ 1\end{pmatrix}\rangle = \sqrt{2}(1+i)$$

8)

$$|u+v|^2 = \langle u+v, u+v\rangle = |u|^2 + |v|^2 + 2\langle u, v\rangle$$

$$\xrightarrow{C.S.} \text{(i)}\quad |u+v|^2 \leq |u|^2 + |v|^2 + 2|u||v| = (|u|+|v|)^2$$

$$\text{(ii)}\quad |u+v|^2 \geq |u|^2 + |v|^2 - 2|u||v| = (|u|-|v|)^2 .$$

9a)

$$\langle e_n, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} e^{inx}\, dx = 1$$

$$\langle e_n, e_{n+l}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} e^{i(n+l)x} = \frac{1}{2\pi}\int_0^{2\pi} e^{ilx}\, dx$$

$l \neq 0$

$$= \frac{1}{2\pi} \left.\frac{e^{ilx}}{il}\right|_0^{2\pi} = \frac{1}{2\pi i l}\left(e^{2\pi l i} - 1\right) \underset{\uparrow \atop l \in \mathbb{Z}}{=} 0$$

9 b)

$$\cos(ax) = \frac{1}{2}\left(e_a(x) + e_{-a}(x)\right)$$

$$\sin(bx) = \frac{1}{2i}\left(e_b(x) - e_{-b}(x)\right)$$

$$\rightsquigarrow \quad I_1 = 2\pi \left(\frac{1}{2\pi} \int_0^{2\pi} \overset{*}{\cos}(ax)\cos(bx)\,dx\right) = 2\pi \langle x \mapsto \cos(ax), x \mapsto \cos(bx)\rangle$$

$$= \frac{2\pi}{4} \langle e_a + e_{-a}, e_b + e_{-b}\rangle = \pi\,\delta_{ab} \qquad (a, b \in \mathbb{N})$$

$$I_2 = \frac{2\pi}{4i} \langle e_a + e_{-a}, e_b - e_{-b}\rangle = 0$$

$$I_3 = \frac{2\pi}{4} \langle e_a - e_{-a}, e_b - e_{-b}\rangle = \pi\,\delta_{ab}$$

10) $\langle .,. \rangle_E$ offenbar symmetrisch, positiv (da $m_i > 0$) und linear, $\rightsquigarrow$ $\langle .,. \rangle_E$ Skalarprodukt;

$$\left.\begin{aligned}
|v|_E^2 &= \langle v, v\rangle_E = \frac{1}{2}\sum_i m_i v_i^2 = E \\
|v + u_b|_E^2 &= \frac{1}{2}\sum_i m_i (v_i + v)^2 = E' \\
|u_b|_E^2 &= \frac{1}{2}\sum_i m_i v^2 = \frac{M}{2} v^2
\end{aligned}\right\} \quad (*)$$

Δ-Ugl.: $\quad \big|\,|v|_E - |u_b|_E\big| \leq |v + u_b|_E \leq |v|_E + |u_b|_E$

$$\overset{(*)}{\Longleftrightarrow} \quad \left|\sqrt{E} - \sqrt{\frac{M}{2} v^2}\right| \leq \sqrt{E'} \leq \sqrt{E} + \sqrt{\frac{M}{2} v^2}$$