Wdhlg:

$$\boxed{\begin{array}{l}\text{Determinante (= n-dim. orient. Volumen) def. durch}\\ (S1)\quad \det(\dots, \lambda A_i, \dots) = \lambda \det(\dots, A_i, \dots) \quad \text{(Skalierung)}\\ (S2)\quad \det(\dots, A_i, \dots, A_j \dots) = \det(\dots, A_i + \lambda A_j, \dots, A_j \dots) \quad \text{(Scherinvarianz)}\\ (S3)\quad \det(\mathbb{1}_n) = 1 \quad \text{(Normierung)}\end{array}}$$

$\Rightarrow$

- Spalten-Linearität:

$$\det(A_1, \dots, A_i + b, \dots) = \det(A_1, \dots, A_i, \dots) + \det(A_1, \dots, b, \dots)$$

- Antisymmetrie:

$$\det(\dots, A_i, \dots, A_j \dots) = \ominus \det(\dots, A_j, \dots, A_i, \dots)$$

- $A_1, \dots, A_n$ linear unabhängig $\Leftrightarrow$ $\det(A) \neq 0$ $\Leftrightarrow$ $A$ bijektiv

- Leibniz-Formel

$$\det(A) = \sum_\sigma \operatorname{sgn}(\sigma) A_{\sigma_1 1} A_{\sigma_2 2} \cdots A_{\sigma_n n}$$

$$\operatorname{sgn}(\sigma) = \det(e_{\sigma_1}, e_{\sigma_2}, \dots, e_{\sigma_n}) = \begin{cases} +1 & : \sigma \text{ "gerade"} \\ -1 & : \sigma \text{ "ungerade"} \end{cases}$$

mit Levi-Civita-Symbol

$$\boxed{\varepsilon_{i_1 i_2 \dots i_n} := \det(e_{i_1}, e_{i_2}, \dots, e_{i_n})}$$

$$\det(A) = \sum_{i_1, \dots, i_n} \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \cdots A_{i_n n}$$

Für $\underline{n=3}$: $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$

$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$

sonst $\varepsilon_{ijk} = 0$

$$\rightarrow \det A = \sum_{ijk} \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

$$= \langle A_1, A_2 \times A_3 \rangle$$

$$(a \times b)_i = \sum_{jk} \varepsilon_{ijk} a_j b_k$$

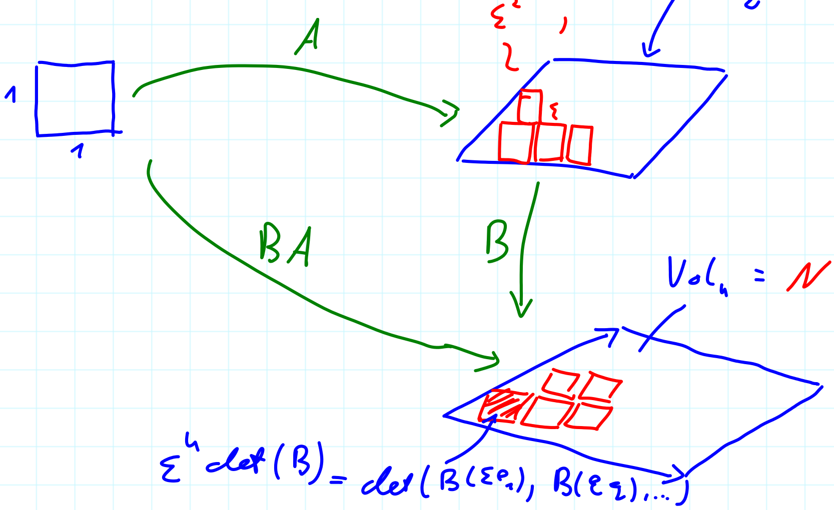
• Produkt-Satz

$$\boxed{\det(BA) = \det(B) \cdot \det(A)}$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

Für geometrisch:

$$N = \frac{Vol_n}{\varepsilon^n} = \frac{1}{\varepsilon^n} \det(A) \quad (*)$$




$$Vol_n = N \varepsilon^n \det(B)$$

$$= \det(A) \det(B)$$

$$\overset{!}{=} \det(BA)$$

$$\varepsilon^n \det(B) = \det(B(\varepsilon e_1), B(\varepsilon e_2), \ldots)$$

formal: 1. Fall: $\det(B) = 0 \rightarrow B$ nicht injektiv

$\rightarrow BA$ nicht injektiv $\rightarrow \det(BA) = 0$ ✓

2. Fall: $\det(B) \neq 0$

Def: $\mathcal{F}: M(n\times n, \mathbb{K}) \longrightarrow \mathbb{K}$

$X = (X_1, X_2, \dots, X_n) \longmapsto \mathcal{F}(X) := \det(BX)$

(Spaltenvektoren: $X_i \in \mathbb{K}^n$) $= \det(BX_1, BX_2, \dots, BX_n)$

- $\mathcal{F}(\dots, \lambda X_i, \dots) \overset{!}{=} \lambda \mathcal{F}(\dots, X_i, \dots)$ $\quad$ ($F1 \hat{=} D1$)

- $\mathcal{F}(\dots, X_i, \dots, X_j \dots) = \mathcal{F}(\dots, X_i + \lambda X_j, \dots, X_j \dots)$ $\quad$ ($F2 \hat{=} D2$)

$\ulcorner$ l.S. $= \det(BX_1, \dots, BX_i, \dots, BX_j \dots) \overset{D2}{=} \det(\dots, \underbrace{BX_i + \lambda BX_j}_{B(X_i + \lambda X_j)}, \dots)$

$= \mathcal{F}(\dots, X_i + \lambda X_j, \dots)$ ✓ $\lrcorner$

- $\mathcal{F}(\mathbb{1}) = \det(B) \neq 0$

$(F1 - F2) \longrightarrow$ Eigenschaften $(F4 - F6)$

$\hat{=} (D4 - D6)$

$\rightarrow$ modifizierte Leibnitz-Formel:

$$\det(BA) \overset{!}{=} \mathcal{F}(A) = \underbrace{\left(\sum_\sigma \operatorname{sgn}(\sigma) A_{\sigma_1 1} A_{\sigma_2 2} \cdots A_{\sigma_n n}\right)}_{\det(A)} \underbrace{\mathcal{F}(e_1, \dots, e_n)}_{= \det(B)}$$

$\blacksquare$

(D9) $\boxed{\det(A) = \det(A^T)}$ (bzg. Determinante Zeilen $\hat{=}$ Spalten)

$$\ulcorner \quad \det(A) = \sum_\sigma \underbrace{\operatorname{sgn}(\sigma)}_{\operatorname{sgn}(\sigma^{-1})} \underbrace{A_{\sigma_1 1} A_{\sigma_2 2} \cdots A_{\sigma_n n}}_{A_{1\sigma_1^{-1}} A_{2\sigma_2^{-1}} \cdots A_{n\sigma_n^{-1}}}$$

$$\underset{\rho = \sigma^{-1}}{\overset{\nearrow}{=}} \sum_\rho \operatorname{sgn}(\rho) \underbrace{A_{1\rho_1}}_{=(A^T)_{\rho_1 1}} \underbrace{A_{2\rho_2}}_{=(A^T)_{\rho_2 2}} \cdots \underbrace{A_{n\rho_n}}_{=(A^T)_{\rho_n n}} = \det(A^T)$$

$\lrcorner$

## Berechnung von Matrizen:

$n=2$ : $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$n=3$ : $\det(A_1, A_2, A_3) = \langle A_1, A_2 \times A_3 \rangle$

$= $ "Sarrus" ✓

$n \geq 4$ :

$$\det\begin{pmatrix} d_1 & & & & 0 \\ \vdots & 1 & & & \\ \vdots & & 1 & & \\ \vdots & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} = d_1 \det\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = d_1$$

$d_1 e_1$

$$\det\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{pmatrix} \overset{D1}{=} d_1 d_2 \cdots d_n$$

$$\det\begin{pmatrix} d_1 & a_{12} & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \overset{D2}{=} \det\begin{pmatrix} d_1 & 0 & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = d_1 \cdots d_n$$

$-\frac{a_{12}}{d_1} A_1$

$$\det\begin{pmatrix} d_1 & a_{12} & a_{13} \cdots a_{1n} \\ & d_2 & a_{23} \cdots a_{2n} \\ 0 & & \ddots \\ & & & d_n \end{pmatrix} \overset{D2}{=} \det\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} = d_1 d_2 \cdots d_n$$

allgemeine Matrix $A$:

$$
\det\begin{pmatrix}
d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\
b_{21} & d_2 & a_{23} & & a_{2n} \\
b_{31} & b_{32} & \ddots & & \vdots \\
\vdots & & & & \\
b_{n1} & b_{n2} & \cdots & b_{n(n-1)} & d_n
\end{pmatrix}
$$

$-\frac{b_{n1}}{d_n} A_n$ $\quad + \quad +$

$-\frac{b_{n2}}{d_n} A_n$

$$
D2 \; = \det\begin{pmatrix}
d_1' & & & & \\
\cdot & d_2' & & & \\
\cdot & \cdot & \ddots & & \\
\cdot & \cdot & \cdot & d_{n-1}' & \\
0 & 0 & \cdots & 0 & d_n
\end{pmatrix}
$$

$\vdots$

$$
D2 \; = \det\begin{pmatrix}
\tilde{d}_1 & & (\tilde{a}_{ij}) & \\
& \tilde{d}_2 & & \\
& 0 & \ddots & \\
& & & \tilde{d}_n
\end{pmatrix} = \tilde{d}_1 \tilde{d}_2 \cdots \tilde{d}_n \quad !
$$

"Gauß - Verfahren"

$\rightsquigarrow O(n^3)$ Operationen !

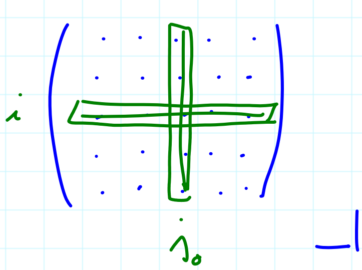
Leibniz - Formel : $\quad n!$ Operationen $\left( n! \approx \left(\frac{n}{e}\right)^n \right)$

## Laplace - Entwicklung

wähle $j_0 \in \{1, \ldots, n\}$ beliebig aber fest ;

$$
\underbrace{\det(A)}_{n \times n \text{ Matrix}} = \sum_{i=1}^{n} (-1)^{i+j_0} A_{ij_0} \underbrace{\det\left( [A]_{ij_0} \right)}_{(n-1) \times (n-1) \text{ Matrix}}
$$

mit $[A]_{ij_0}$ = $(n-1)\times(n-1)$ Untermatrix,
erhalten aus $A$ durch Weglassen
der $i$-ten Zeile und $j_0$-ten Spalte




**Bsp.**

1) $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{j_0=1}{=} (-1)^{1+1}\, a \underbrace{\det\begin{pmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{pmatrix}}_{\underbrace{\det d}_{= d}} + (-1)^{2+1}\, c \underbrace{\det\begin{pmatrix} \cancel{a} & b \\ \cancel{c} & \cancel{d} \end{pmatrix}}_{= b}$$

$$= ad - cb \quad ✓$$

2) $$\det\begin{pmatrix} \cos\varphi & -\sin\varphi & \cancel{0} \\ \sin\varphi & \cos\varphi & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} \end{pmatrix} \underset{\substack{j_0 \\ i=3}}{=} (-1)^{1+3} \cdot 1 \cdot \underbrace{\det\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{=1}$$

$$= 1.$$

**Beweis-Idee:** $j_0 \in \{1, \dots, n\}$ fest gewählt.

$$\det(A_1, \dots, A_{j_0}, \dots A_n) = (-1)^{j_0-1} \det(A_{j_0}, A_1, A_2 \dots A_n)$$

$$= (-1)^{j_0-1} \cdot \sum_{i=1}^{n} A_{ij_0} \det(e_i, A_1, \dots, A_n)$$

$$\left( A_{j_0} = \sum_{i=1}^{n} A_{ij_0} e_i \right.$$

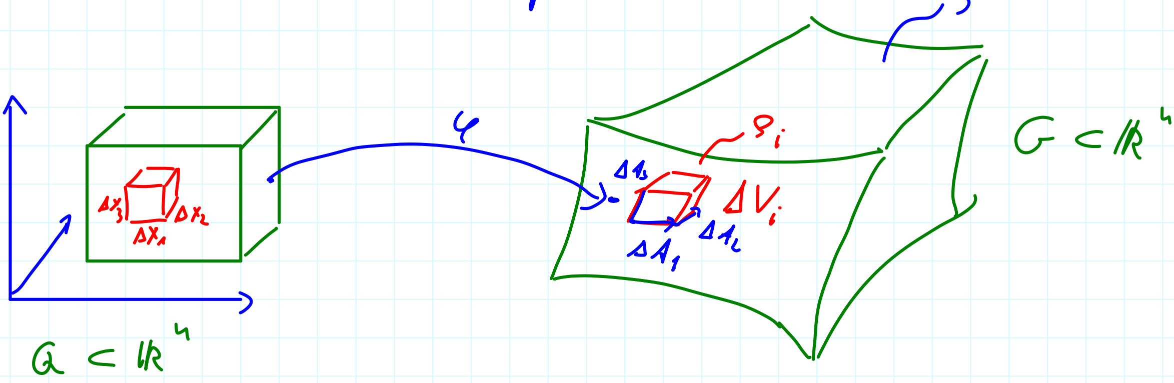
$$\det(e_i, A_1, \dots) = \det\begin{array}{c}\begin{pmatrix} 0 & A_{11} & A_{12} & \cdots \\ 0 & & & \\ \vdots & & & \\ 0 & & & \\ 1 & A_{i1} & A_{i2} & \cdots \\ 0 & & & \\ 0 & & & \end{pmatrix}\end{array} \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \end{array}$$

$$= (-1)^{i-1} \underbrace{\det\begin{pmatrix} 1 & A_{i1} \;\; A_{i2} \\ 0 & \\ \vdots & [A]_{ij_0} \\ 0 & \end{pmatrix}}_{\det([A]_{ij_0})}$$

$$\rightarrow \det(A) = \sum_{i=1}^{n} (-1)^{i+j_0 \cancel{-2}} A_{ij_0} \det([A]_{ij_0}) \quad \checkmark$$

<u>(Ex)$^2$ – hms.</u>

## n-dimensionale Integrale




$$G = \varphi(Q) := \{ \varphi(x) \mid x \in Q \}$$

**kurze, Parametrisierung**

$$\varphi : Q \longrightarrow G$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \end{pmatrix}$$

Bsp:

$$Q = [0,R] \times [0,\pi] \times [0,2\pi] \longrightarrow K_R$$

$$(r, \vartheta, \varphi) \longmapsto \varphi(r,\vartheta,\varphi) = r \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

Skalarfeld $\varrho : \mathbb{R}^n \to \mathbb{R}$

**Integral von $\varrho$ über Gebiet $G$:**

$$\int_G \varrho \, dV := \sum_i \varrho_i \, \Delta V_i$$

$$\Delta V_i = \det(\Delta A_1, \Delta A_2, \dots, \Delta A_n)$$

$$\Delta A_1 = \frac{\partial \varphi(x)}{\partial x_1} \cdot \Delta x_1, \qquad \Delta A_2 = \frac{\partial \varphi}{\partial x_2}(x)\, \Delta x_2, \quad \dots$$

$$\rightarrow \Delta V_1 = \det\Big( \underbrace{\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \dots, \frac{\partial \varphi}{\partial x_n}}_{\partial_x \varphi} \Big) \Delta x_1 \Delta x_2 \cdots \Delta x_n$$

mit Jacobi-Matrix

$$J_x\varphi := \left( \frac{\partial \varphi}{\partial x_1}(x), \frac{\partial \varphi}{\partial x_2}(x), \dots \right)$$

$$= \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \dots \\ \frac{\partial \varphi_2}{\partial x_2} & & \\ \vdots & & \end{pmatrix}$$

$$\rightarrow \boxed{\begin{aligned} \int_G \rho \, dV &:= \sum_i \rho(\varphi(x)) \left| \det(J_{x_i}\varphi) \right| \underline{\Delta x_1} \cdots \underline{\Delta x_n} \\ &:= \int_Q \rho(\varphi(x)) \left| \det(J_x\varphi) \right| dx_1 \, dx_2 \cdots dx_n \end{aligned}}$$