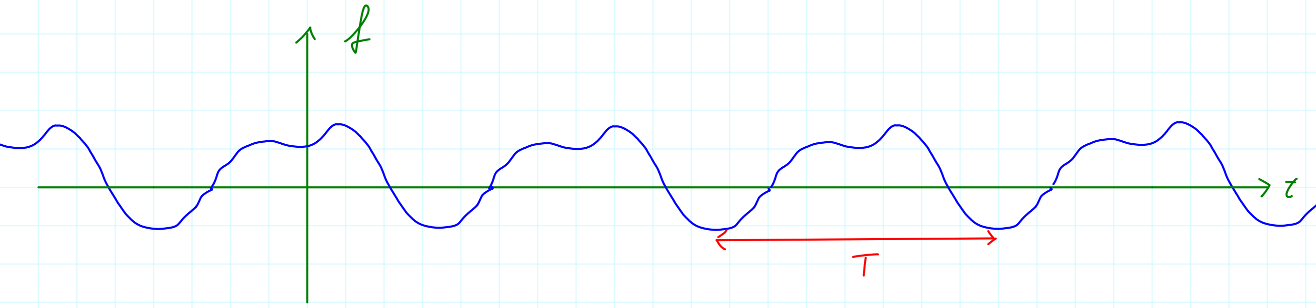
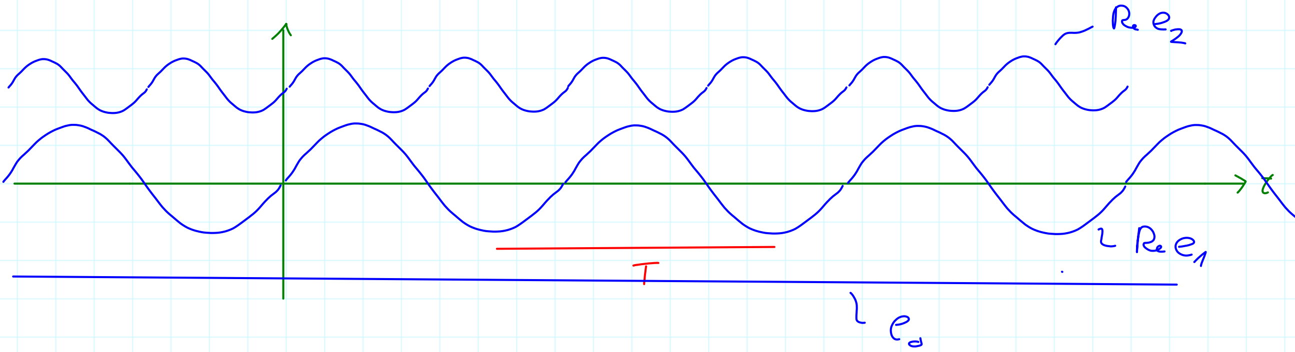
Wdhlg. Fourierreihe einer T-per. Fkt. $f(t)$ $\quad (= f(t+T))$



= Darstellung als Linearkombination der Fkt.en

$$e_n : t \longmapsto e^{i\omega_n t}, \qquad n \in \mathbb{Z}, \quad \omega_n = \frac{2\pi}{T} n$$



d.h. $\boxed{f(t) = \sum_{n\in\mathbb{Z}} f_n e^{i\omega_n t}}$ • Fourierreihe der T-per. Fkt $f(t)$

n-ter Fourierkoeffizient • $\omega_n = \frac{2\pi}{T} \cdot n$

$$\boxed{f = \sum_{n\in\mathbb{Z}} f_n e_n} \qquad f_n = ?$$

$V \ni v = \sum_{i=1}^{d} v_i e_i$ — ONB $e_1, \dots, e_d$

i-te Komponenten von v bzgl.

$\rightarrow \quad v_i = \langle e_i, v \rangle$!

bzgl. Skalarprodukt $\langle f, g\rangle := \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\, g(t)\, dt$

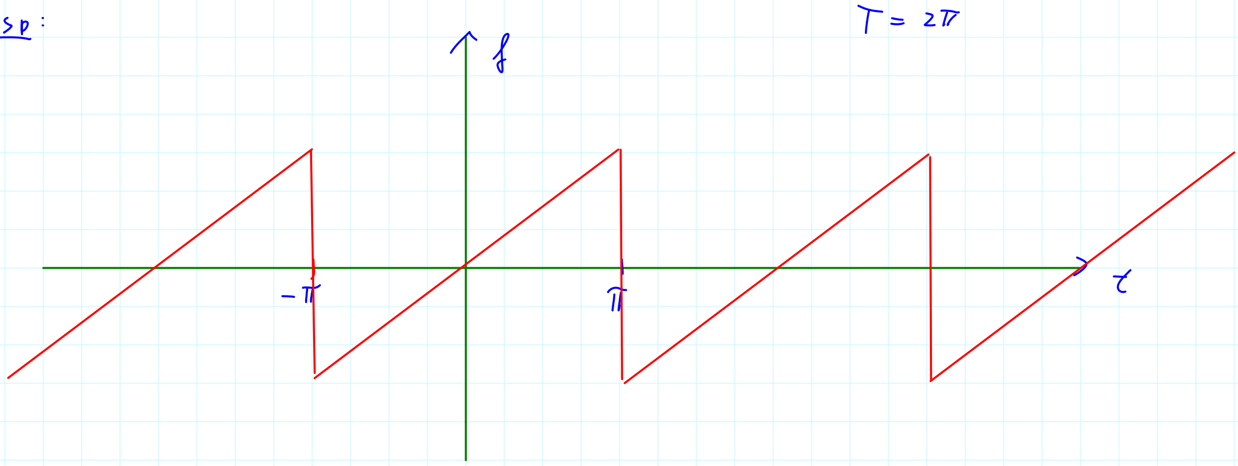
$\{e_n\}_{n\in\mathbb{Z}}$ <u>orthonormales</u> Funktionensystem

$$\rightarrow \boxed{f_n = \langle e_n, f\rangle = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}} \quad ✓$$

$\omega_n = \frac{2\pi}{T} n$

<u>Bsp</u>:

$T = 2\pi$



2π-per. Fkt $f(t) = t \;:\; t\in\,]-\pi, \pi]$

$\Rightarrow$ Fourierreihe: $f(t) = \sum_{n\in\mathbb{Z}} f_n\, e^{i\omega_n t}$

$\omega_n = \frac{2\pi}{T} n = n$

mit Fourierkoeff.:

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\, e^{-int}\, dt = \begin{cases} 0 & : n = 0 \\ i\,\frac{(-1)^n}{n} & : n \neq 0 \end{cases}$$

für $n \neq 0$:

$$f_n = \frac{1}{2\pi}\left\{ \frac{t\, e^{-int}}{-in}\Bigg|_{-\pi}^{+\pi} + \underbrace{\int_{-\pi}^{+\pi} \frac{e^{-int}}{in}}_{0} \right\}$$

P.I.

$$= \frac{1}{2\cancel{\pi}} \left\{ \frac{\cancel{\pi}\, e^{-i\pi n}}{-in} + \frac{\cancel{\pi}\, e^{+i\pi n}}{-in} \right\} = \frac{1}{2} \, \frac{1}{-in} \left((-1)^n + (-1)^n \right)$$

$$f_n = i \, \frac{(-1)^n}{n}$$

$$\rightarrow \quad f(t) = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} i \, \frac{(-1)^n}{n} \, e^{int} = \sum_{n=1}^{\infty} \frac{i \, (-1)^n}{n} \underbrace{\left(e^{int} - e^{-int} \right)}_{= \, 2i \sin(nt)}$$

$$f(t) = 2 \sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt)$$

$$= 2 \sin t - \sin(2t) + \frac{2}{3} \sin(3t) - \frac{1}{2} \sin(4t) + \ldots - \ldots$$

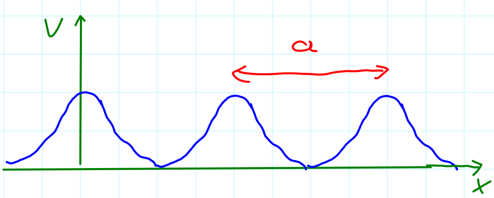
Notation:

f Fkt der Zeit t: $f(t)$ Periode T $\rightarrow$ $\omega = \frac{2\pi}{T}$

$\omega_n = \omega \cdot n = \frac{2\pi}{T} n$

f Fkt der Koord. x: $f(x)$ Periode a $\rightarrow$ $k = \frac{2\pi}{a}$

$k_n = k \cdot n = \frac{2\pi}{a} n$



Wellenzahlen

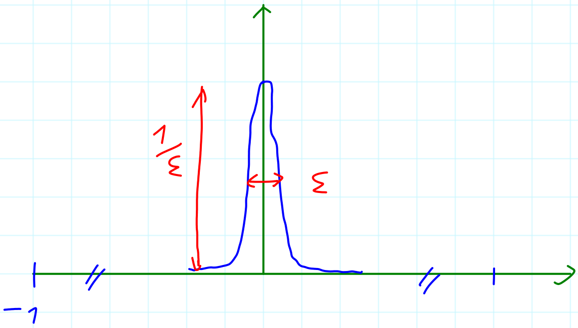
$$\rightarrow \quad f(x) = \sum_{n \in \mathbb{Z}} f_n \, e^{i k_n x}$$

$$f_n = \frac{1}{a} \int_{-a/2}^{+a/2} f(x) \, e^{-i k_n x} \, dx$$

Exkurs: Dirac-δ-Funktion ($\to$ Dirac-δ-Distribution)

$\Rightarrow$ für $0 < \varepsilon \ll 1$:

nicht-negative, bel. oft. diff.bare Fkt $\delta_\varepsilon : \mathbb{R} \to \mathbb{R}$



mit Eigenschaft:
$$\int_a^b \delta_\varepsilon(x)\,dx = \begin{cases} 1 & : \; 0 \in [a,b] \\ 0 & : \; 0 \notin [a,b] \end{cases}$$

im Grenzfall $\varepsilon \to 0$

$\rightsquigarrow$ stetige Fkt f :

$$\int_{-\infty}^{+\infty} f(x)\,\delta_\varepsilon(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = f(0)$$

im Grenzfall $\varepsilon \to 0$

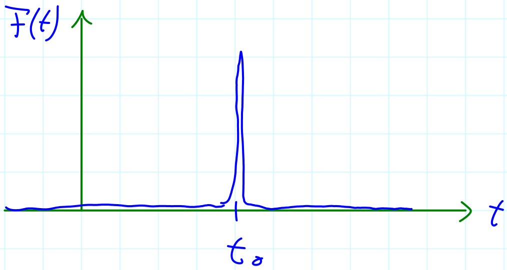
$\to$ Dirac-δ-Fkt: "$\delta(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x)$"

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)} \qquad \sum_{n\in\mathbb{Z}} f_n\,\delta_{n,0} = f_0$$

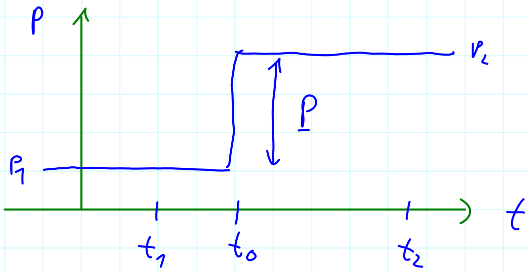
$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)} \qquad \sum_{n\in\mathbb{Z}} f_n\,\delta_{n,j} = f_j$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(\lambda(x-a))\,dx = \frac{1}{\lambda} f(a)}$$

Bsp: 1D "Kraftstoß": $F(t) \overset{!}{=} P\,\delta(t-t_0)$



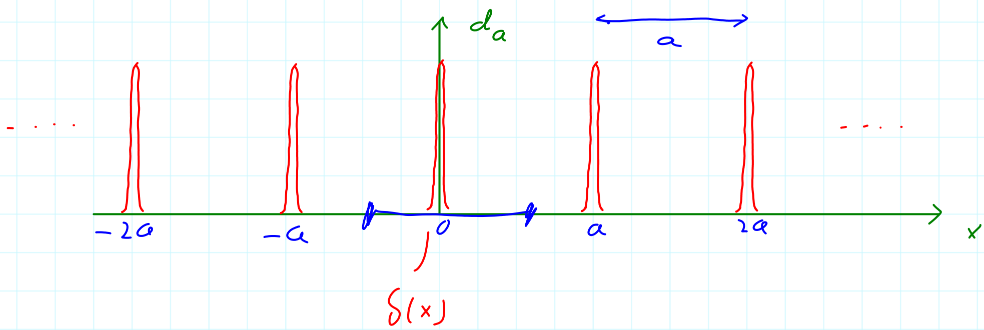
$\dot{p}(t) = F(t)$



$$p_2 - p_1 = p(t_2) - p(t_1) = \int_{t_1}^{t_2} \dot{p}(t)\,dt = \int_{t_1}^{t_2} F(t)\,dt = \underbrace{\int_{t_1}^{t_2} P\,\delta(t-t_0)\,dt}_{= P}$$

Fourierreihe des "Dirac-Kamms"

$$\boxed{d_a(x) := \sum_{n\in\mathbb{Z}} \delta(x-na)}$$



für $a = 1$: $$d(x) = \sum_{n \in \mathbb{Z}} \delta(x-n)$$

$\Rightarrow$ Fourierreihe $$d(x) = \sum_{n \in \mathbb{Z}} d_n \, e^{i k_n x} \quad , k_n = 2\pi n$$

$$= \sum_{n \in \mathbb{Z}} d_n \, e^{2\pi i n x}$$

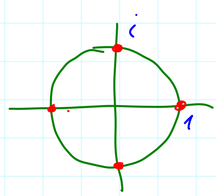
Fourierkoeff.: $$d_n = \int_{-1/2}^{+1/2} d(x) \, e^{-i 2\pi n x} \, dx = 1$$

$$\rightarrow \quad \boxed{d(x) = \sum_{n \in \mathbb{Z}} \delta(x-n) \overset{!}{=} \sum_{n \in \mathbb{Z}} e^{2\pi i n x}}$$

$x = 0$: "∞" : "∞" $= \sum_{n \in \mathbb{Z}} 1$ ✓

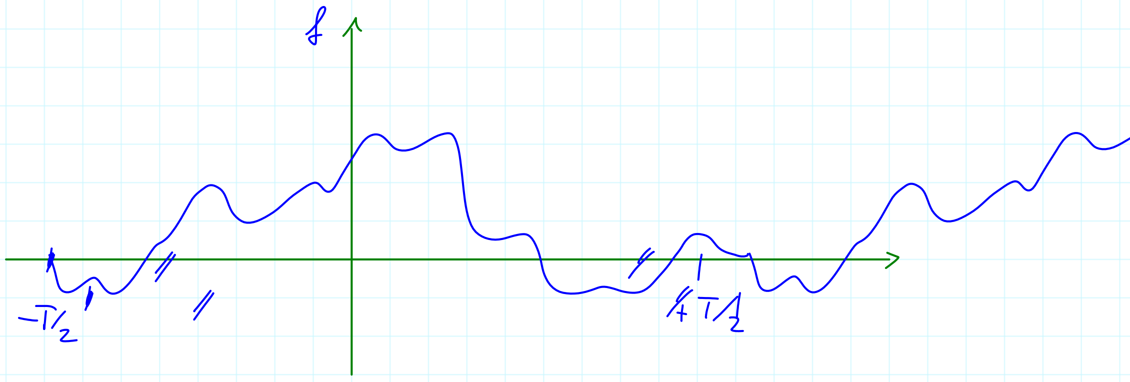
$x = 1/2$: 0 : $\sum_{n \in \mathbb{Z}} e^{i\pi n} = \sum_{n \in \mathbb{Z}} (-1)^n \overset{!}{=} 0$! ✓

$x = 1/4$ 0 : $\sum_{n \in \mathbb{Z}} e^{i \frac{\pi}{2} n} = \sum_{n \in \mathbb{Z}} i^n = 0$ ✓



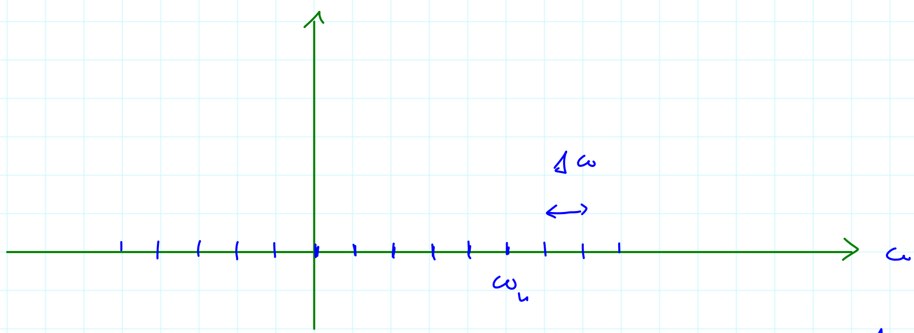
Fouriertransformation

Trick: bel. Fkt $f(t)$ $\hat{=}$ T-per. Fkt $f(t)$ im Grenzfall $T \to \infty$



$f(t)$, $\omega_n = \frac{2\pi}{T} n = \Delta\omega \cdot n$

mit $\Delta\omega = \frac{2\pi}{T}$ $\qquad \Delta\omega \xrightarrow[T\to\infty]{} 0$



$\Delta\omega = 2\pi/T$

$$f(t) = \sum_{n\in\mathbb{Z}} \underbrace{\frac{1}{\Delta\omega} f_n}_{\hat{f}(\omega)} e^{i\omega_n t} \underbrace{\Delta\omega}_{d\omega} = \sum_{n\in\mathbb{Z}} \frac{1}{\Delta\omega} f_{\frac{\omega_n}{\Delta\omega}} e^{i\omega_n t} \Delta\omega$$

$\sum_{n\in\mathbb{Z}} \to \int_{-\infty}^{\infty}$

$\omega_n = \Delta\omega \cdot n$

$$\underset{\Delta\omega\to 0}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi} \underbrace{T f_{\frac{\omega}{\Delta\omega}}}_{=:\hat{f}(\omega)} e^{i\omega t} \, d\omega$$

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_n t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\Delta\omega \cdot n\, t}\, dt$$

$$\hat{f}(\omega) = T\, f_{\frac{\omega}{\Delta\omega}} = \int_{-T/2}^{T/2} f(t)\, e^{-i\cancel{\Delta\omega}\frac{\omega}{\cancel{\Delta\omega}} t}\, dt$$

$T \to \infty$

Fouriertransformierte von $f(t)$:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

inverse F.T.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{+i\omega t}\, d\omega$$