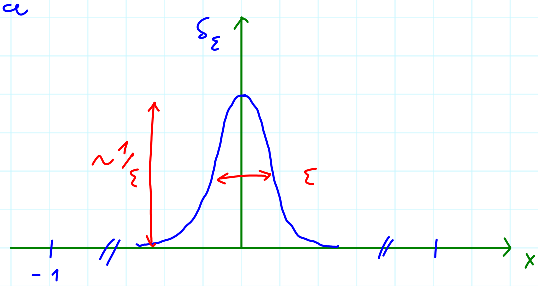
Wdhlg:

Dirac-δ_ε-Funktion für $0 < \varepsilon << 1$: $\delta_\varepsilon : \mathbb{R} \to [0,\infty[$

bel. oft diffbar. mit Eigenschaft:

$$\int_a^b \delta_\varepsilon(x)\,dx \underset{\varepsilon \to 0}{=} \begin{cases} 1 : & 0 \in]a,b[\\ 0 : & 0 \notin [a,b] \end{cases}$$

d.h.



konkret: $\delta_\varepsilon = \varepsilon$-Gauß-Fkt:

$$\boxed{\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon^2}}\, e^{-x^2/2\varepsilon^2}}$$

$\to$ Dirac-δ-Funktion: $\delta(x) = \lim\limits_{\varepsilon \to 0} \delta_\varepsilon(x)$

2> Eigenschaften ("Rechenregeln") der δ-Fkt:

- $\displaystyle\int_a^b f(x)\,\delta(x)\,dx = f(0)$ (für $a < 0 < b$ und f stetig in $x = 0$)

- $\displaystyle\int_a^b f(x)\,\delta(x - x_0)\,dx = f(x_0)$ ($a < x_0 < b$)

$$\boxed{\bullet \quad \int_a^b f(x) \frac{\partial}{\partial x} \delta(x-x_0)\, dx = -\frac{\partial f}{\partial x}(x_0)}$$

$$\left[\int_a^b f(x) \frac{\partial}{\partial x} \delta(x-x_0)\, dx \underset{\text{P.I.}}{=} \underbrace{f(x)\,\delta(x-x_0)\Big|_a^b}_{=0} - \underbrace{\int_a^b \frac{\partial f}{\partial x}(x)\, \delta(x-x_0)\, dx}_{=\frac{\partial f}{\partial x}(x_0)} = -\frac{\partial f}{\partial x}(x_0) \right]$$

$\Rightarrow$ n-fache Anwendung:

$$\boxed{\bullet \quad \int_a^b f(x) \frac{\partial^n}{\partial x^n} \delta(x-x_0)\, dx = (-1)^n \frac{\partial^n f}{\partial x^n}(x_0)}$$

<u>Substitutionregel:</u> $g: [\alpha, \beta] \to [a, b]$,

invertierbar, Nullstelle bei $x_0 \in]\alpha, \beta[$

$$\to \quad \boxed{\int_\alpha^\beta f(x)\, \delta(g(x))\, dx = f(x_0) \frac{1}{|g'(x_0)|}}$$

$\Big[$ g monoton steigend (fallend):

$$\int_\alpha^\beta f(x)\, \delta(g(x))\, dx \underset{x = g^{-1}(u)}{=} (-) \int_a^b f(g^{-1}(u))\, \delta(u)\, \underbrace{(g^{-1})'(u)}_{= \frac{1}{g'(g^{-1}(u))}}\, du$$

$$= \int_a^b \frac{f(g^{-1}(u))}{|g'(g^{-1}(u))|}\, \delta(u)\, du \underset{g^{-1}(0) = x_0}{=} \frac{f(x_0)}{|g'(x_0)|} \Big]$$

Fouriertransformation

Def.

Die **Fouriertransformierte** der Fkt. $f: \mathbb{R} \to \mathbb{C}$, $x \mapsto f(x)$

ist die **Funktion** $\hat{f}: \mathbb{R} \to \mathbb{C}$, $k \mapsto \hat{f}(k)$

def. durch

$$\boxed{\hat{f}(k) := \int_{-\infty}^{\infty} f(x)\, e^{-ikx} \frac{dx}{\sqrt{2\pi}}}$$

← Fouriertrafo von $f(x)$ auf $\hat{f}(k)$

Satz

es gilt:

$$\boxed{f(x) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{+ikx} \frac{dk}{\sqrt{2\pi}}}$$

← inverse Fouriertrafo von $\hat{f}(k)$ auf $f(x)$

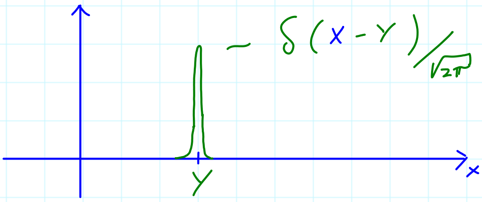
d.h.: Fouriertransformation ist Abbildung zwischen Funktionen

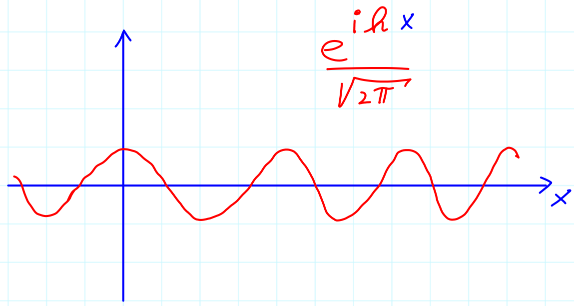
$$\boxed{\begin{aligned} \mathcal{F} &: f \longmapsto \mathcal{F}[f] := \hat{f} \\ \mathcal{F}^{-1} &: \hat{f} \longmapsto \mathcal{F}^{-1}[\hat{f}] = f \end{aligned}}$$

also

- $\hat{f} = \mathcal{F}[f]$,
- $\hat{f}(k) = \mathcal{F}[f](k)$,
- $\mathcal{F}^{-1}[\hat{f}] = \mathcal{F}^{-1}[\mathcal{F}[f]] = f$,
- u.s.w.

2) "Teilchen" $\underset{-\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}$ "Welle"





Funktion f:

$$\int_{-\infty}^{\infty} f(y)\, \frac{\delta(x-y)}{\sqrt{2\pi}}\, dy \overset{!}{=} f(x) \overset{!}{=} \int_{-\infty}^{\infty} \hat{f}(k)\, \frac{e^{ikx}}{\sqrt{2\pi}}\, dk$$

Superposition von "Teilchenzuständen"
$\delta(x-y)$

Superposition von "Wellenzuständen"
e^{ikx}

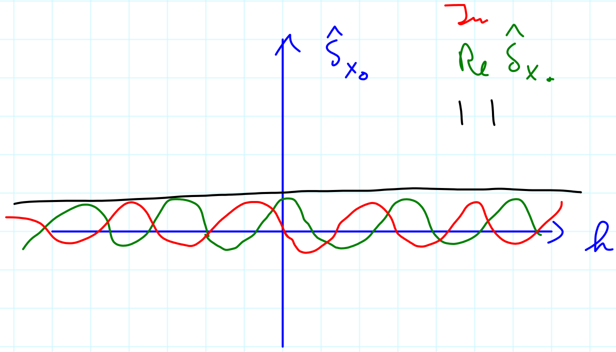
Beispiele:

1) Delta-Funktion:

$$\delta_{x_0}(x) := \delta(x - x_0) \xrightarrow{\tilde{\mathcal{F}}} \hat{\delta}_{x_0}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \delta(x-x_0)\, e^{-ikx}$$

$$= \frac{e^{-ikx_0}}{\sqrt{2\pi}}$$

$$\boxed{\hat{\delta}_{x_0}(k) = \frac{e^{-ikx_0}}{\sqrt{2\pi}}}$$



$$\Rightarrow \quad \delta(x - x_0) \overset{!}{=} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \underbrace{\hat{\delta}_{x_0}(k)}_{\frac{e^{-ikx_0}}{\sqrt{2\pi}}}\, e^{+ikx}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x_0)}$$

d. h. $x_0 = 0$

$$\boxed{\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \overset{!}{=} \delta(x)} \quad (*)$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx - \frac{\varepsilon^2 k^2}{2}} \overset{!}{=} \delta_\varepsilon(x)$$

2) „Ebene Welle"

$$\chi_{k_0}(x) := e^{ik_0 x} \longrightarrow \hat{\chi}_{k_0}(k)$$

$$e^{ik_0 x} = \chi_{k_0}(x) \overset{!}{=} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \underbrace{\hat{\chi}_{k_0}(k)}_{\overset{!}{=}\ \sqrt{2\pi}\,\delta(k-k_0)}\, e^{ikx} = e^{ik_0 x} \quad \checkmark$$

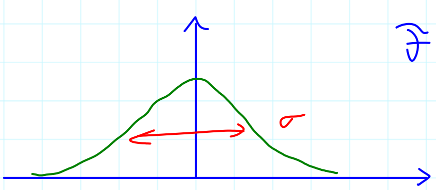
$$\boxed{\hat{\chi}_{k_0}(k) = \sqrt{2\pi}\,\delta(k-k_0)}$$

$$\hat{\chi}_{k_0}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{ik_0 x}\, e^{-ikx} = \sqrt{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ix(k_0-k)}}_{\overset{(*)}{=}\ \delta(k_0-k)}$$

$$= \sqrt{2\pi}\,\delta(k_0-k)$$

3) Gauß-Funktion

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}$$



$$\overset{\mathcal{F}}{\Longrightarrow} \quad \hat{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}\, e^{-ikx}$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma^2} - ikx}$$

Gauß-Integral

Identität

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}}$$

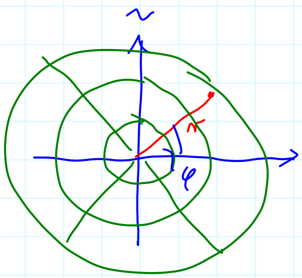
$a \in \mathbb{R}_+$

$b \in \mathbb{C}$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}\,dx \;\underset{x=\frac{u}{\sqrt{a}}}{=}\; \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} \underbrace{e^{-u^2+\frac{b}{\sqrt{a}}u}}_{\parallel \; e^{-\left(u-\frac{b}{2\sqrt{a}}\right)^2+b^2/4a}}\,du$$

$$= \frac{1}{\sqrt{a}}\, e^{b^2/4a} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\,du}_{\parallel \; \sqrt{\pi}} \; * \; = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

(*) $$\left(\int_{-\infty}^{\infty} e^{-u^2}\,du\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\overbrace{(u^2+v^2)}^{r^2}} \underbrace{du\,dv}_{\parallel \; r\,dr\,d\varphi}$$



$r^2 = u^2+v^2$

$$= \int_0^{\infty} dr \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \; r\, e^{-r^2}$$

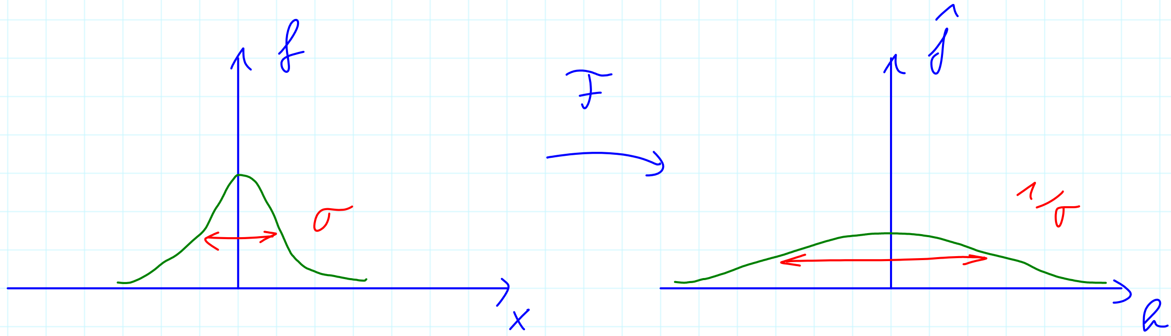
Polarkoordinaten r, φ

$$= 2\pi \int_0^{\infty} dr\; r\, e^{-r^2} = 2\pi \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^{\infty} = \pi$$

$$\rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

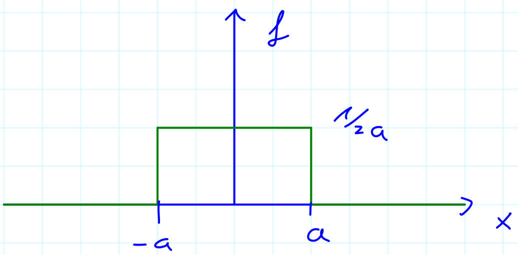
$$\hat{f}(k) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} dx\; e^{-\frac{x^2}{2\sigma^2} - ikx} \;\underset{\substack{a = 1/2\sigma^2 \\ b = ik}}{=}\; \frac{1}{2\pi\sigma}\sqrt{2\pi\sigma^2}\; e^{-\frac{k^2\cdot 2\sigma^2}{4}}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{k^2}{2(1/\sigma^2)}}$$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}} \xrightarrow{\mathcal{F}} \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{k^2}{2(1/\sigma^2)}}$$

4) Kastenfunktion

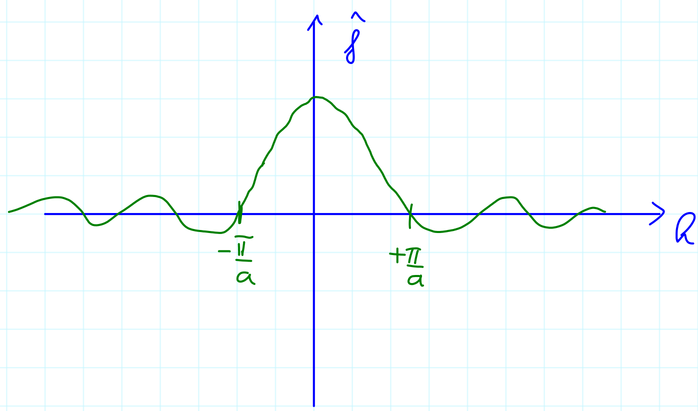


$$f(x) = \begin{cases} 0 : & |x| > a \\ \frac{1}{2a} : & |x| \leq a \end{cases}$$

$$\Rightarrow \quad \hat{f}(k) = \frac{1}{2a}\int_{-a}^{a} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx} = \frac{1}{\sqrt{2\pi}}\,\frac{1}{2a}\left(\frac{e^{-ikx}}{-ik}\Bigg|_{-a}^{a} \right)$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{2a}\left(\frac{e^{ika} - e^{-ika}}{ik} \right) = \frac{1}{\sqrt{2\pi}}\,\frac{\sin(ka)}{ka}$$

$$\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\,\frac{\sin(ka)}{ka}}$$



$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\,\frac{\sin(ka)}{ka}\, e^{ikx} = \begin{cases} 0 : & |x| > a \\ \frac{1}{2a} : & |x| \leq a \end{cases}$$

Eigenschaften der Fouriertransformation:

(i) $\mathcal{F}$ linear: $\widehat{f+g} = \hat{f} + \hat{g}$

$$\widehat{\lambda f} = \lambda \hat{f}$$

(ii) Translation um x_0 $\longrightarrow$ Multiplikation e^{-ikx_0}

$$f(x) \hookrightarrow f_{x_0}(x) := f(x-x_0) \xrightarrow{\mathcal{F}} \hat{f}_{x_0}(k) = e^{-ikx_0}\hat{f}(k)$$

$$f_{x_0}(x) = f(x-x_0) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{f}(k)\, e^{ik(x-x_0)}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \underbrace{e^{-ikx_0}\hat{f}(k)}_{=\hat{f}_{x_0}(k)}\, e^{ikx}$$

(iii) Ableitung $\frac{\partial}{\partial x}$ $\longrightarrow$ Multiplikation mit ik

$$\widehat{\frac{\partial f}{\partial x}}(k) \stackrel{!}{=} ik\,\hat{f}(k)$$

$$\frac{\partial}{\partial x} f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, \underbrace{\frac{\partial e^{ikx}}{\partial x}}_{=ik\,e^{ikx}} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \underbrace{ik\,\hat{f}(k)}_{=\widehat{\frac{\partial f}{\partial x}}(k)}\, e^{ikx}$$