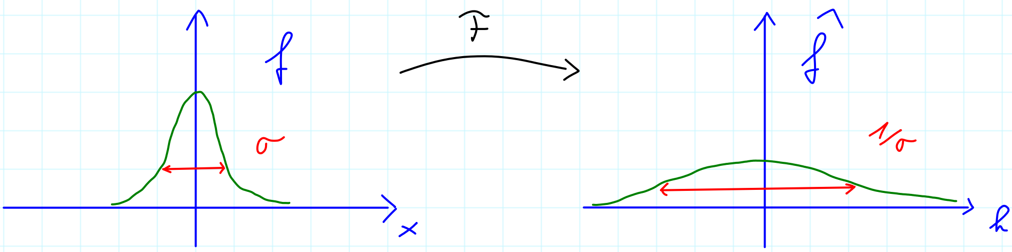
Wdhlg.

$f(x)$	$\hat{f}(k)$
$\delta(x-x_0)$	$\frac{1}{\sqrt{2\pi}}\, e^{-ikx_0}$
$e^{ik_0 x}$	$\sqrt{2\pi}\;\delta(k-k_0)$
$\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}$	$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{k^2}{2(1/\sigma^2)}}$
$f(x) = \begin{cases} 0 : & \lvert x\rvert > a \\ \frac{1}{2a} : & \lvert x\rvert \le a \end{cases}$	$\frac{1}{\sqrt{2\pi}}\, \frac{\sin(ka)}{ka}$



Identität für Gauß-Integral

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2+bx}\, dx = \sqrt{\frac{\pi}{a}}\; e^{b^2/4a}}$$

$a \in \mathbb{R}_+$

$b \in \mathbb{C}$

$$\rightarrow \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx - \frac{\varepsilon^2 k^2}{2}} = \delta_\varepsilon(x)$$

$$\varepsilon = 0 \rightarrow \quad \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} = \delta(x)$$

Eigenschaften der Fouriertransformation

(i) Linearität:

$$\widehat{f+g} = \hat{f} + \hat{g}$$

$$\widehat{\lambda f} = \lambda \hat{f}$$

(ii) Translation um x_0 $\rightarrow$ Multiplikation mit e^{-ikx_0}

$$f_{x_0}(x) := f(x - x_0) \longrightarrow \hat{f}_{x_0}(k) = e^{-ikx_0} \hat{f}(k)$$

(iii) Ableitung $\frac{\partial}{\partial x}$ $\rightarrow$ Multiplikation mit ik:

$$\widehat{\frac{\partial f}{\partial x}}(k) = ik \hat{f}(k)$$

(iv) $\mathcal{F}$ ist unitäre Abbildung auf VR der komplexwertigen Fkten $\mathbb{R} \rightarrow \mathbb{C}$ mit Skalarprodukt

$$\langle f, g \rangle := \int_{-\infty}^{\infty} dx \, f^*(x) \, g(x) :$$

$$\boxed{\begin{aligned} \langle f, g \rangle &= \langle \mathcal{F}[f], \mathcal{F}[g] \rangle = \langle \hat{f}, \hat{g} \rangle \\ |f|^2 &= |\hat{f}|^2 \end{aligned}}$$

(s.u. Plancherel)

$$\int_{-\infty}^{\infty} dx \, |f(x)|^2 \qquad\qquad \int_{-\infty}^{\infty} dk \, |\hat{f}(k)|^2$$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} dx\, f^*(x)\, g(x)$$

$$= \int_{-\infty}^{\infty} dx\, \underbrace{\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{f}^*(k)\, e^{-ikx}}_{f^*(x)}\, \overbrace{\int_{-\infty}^{\infty} \frac{d\tilde{k}}{\sqrt{2\pi}}\, \hat{g}(\tilde{k})\, e^{i\tilde{k}x}}^{g(x)}$$

$$= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\tilde{k}\, \hat{f}^*(k)\, \hat{g}(\tilde{k})\, \underbrace{\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ix(\tilde{k}-k)}}_{=\,\delta(\tilde{k}-k)}$$

$$= \int_{-\infty}^{\infty} dk\, \hat{f}^*(k)\, \hat{g}(k) = \langle \hat{f}, \hat{g}\rangle$$

Anwendung der Fouriertransformation: Wärmeleitung

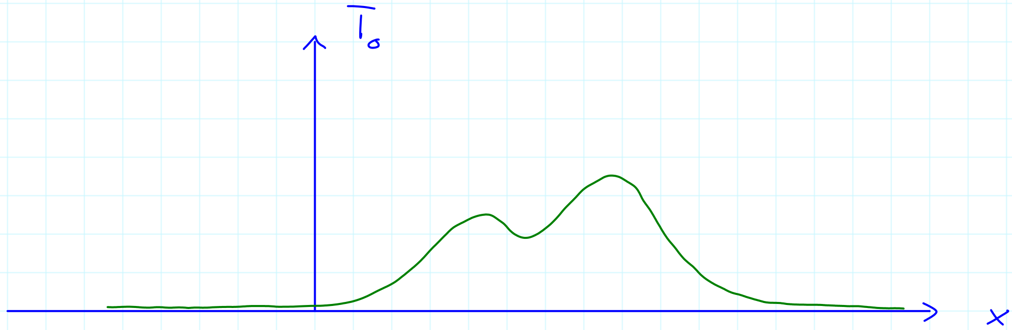
physikalisches Problem:

Wärmeleiter (z.B. Cu-Stab)

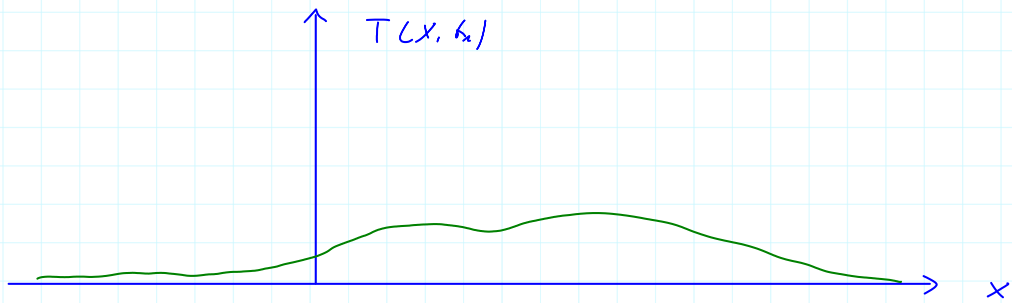
$T(x,t)$: Temperatur am Ort x zur Zeit t

Aufgabe: gegeben Temperatur $T(x,0) = T_0(x)$ zur Zeit $t=0$, wie lautet $T(x,t)$ für Zeit $t>0$?

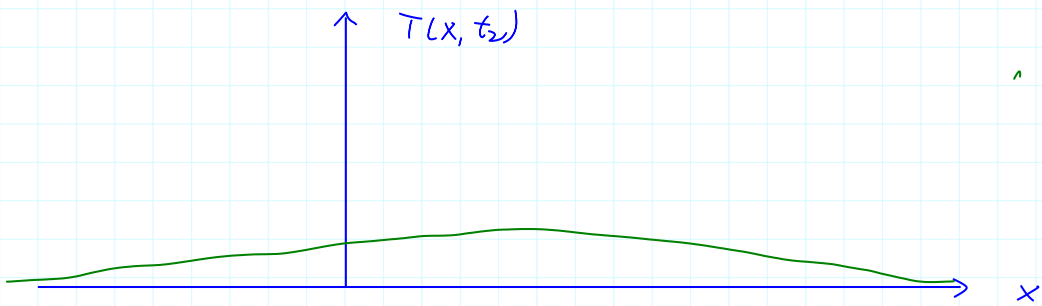
$t=0$:



$t_1 > 0$



$t_2 > t_1$



Wärmeleitungsgesetz:

3D $\quad T(\vec{r}, t)$ genügt der Wärmeleitungsgleichung.

$$\boxed{\frac{\partial T}{\partial t} = a \Delta T}$$

a: Temperaturleitfähigkeit

$$[a] = \frac{\text{Länge}^2}{\text{Zeit}}$$

Lapl.-Op: $\Delta = \operatorname{div} \operatorname{grad} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

1D

$$\boxed{\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}}$$

Diffusionsgl.: $\quad \rho(\vec{r}, t) \rightarrow \frac{\partial \rho}{\partial t} = D \Delta \rho$

Woher?

Wärmestromdichte $\vec{q}(\vec{r},t) \longleftrightarrow T(\vec{r},t)$

$\rightsquigarrow \quad \boxed{\vec{q} = -\lambda \,\mathrm{grad}\, T}$ ✓

λ : Wärmeleitfähigkeit

$$[\lambda] = \frac{W}{m^2 K}$$

Wärmeenergiedichte $u(\vec{r},t) \longleftrightarrow \vec{q}(\vec{r},t)$

$\longleftrightarrow T(\vec{r},t)$

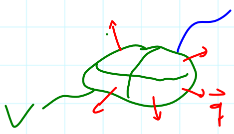
$\rightarrow \quad \boxed{u(\vec{r},t) = \rho \, c \, T(\vec{r},t)}$

Massendichte

spez. Wärmekapazität

Wärmeenergieerhaltung $\rightarrow$ Kontinuitätsgleichung:

$$\boxed{\frac{\partial u}{\partial t} + \mathrm{div}\,\vec{q} = 0}$$ ✓



$$W_V = \int_V u \, d^3\vec{r}$$

$$I_V = \int_{\partial V} \vec{q}\, d\vec{f} \underset{\text{s.v.G.}}{=} \int_V \mathrm{div}\,\vec{q}\, d^3\vec{r}$$

$$\frac{\partial W_V}{\partial t} \overset{!}{=} -I_V = -\int_V \mathrm{div}\,\vec{q}\, d^3\vec{r}$$

$$\| $$

$$\int_V \frac{\partial u}{\partial t}\, d^3\vec{r} \quad \rightsquigarrow \quad \int_V \underbrace{\left(\frac{\partial u}{\partial t} + \mathrm{div}\,\vec{q}\right)}_{=0\,!!} d^3\vec{r} = 0$$

da V beliebig!

$$\rightarrow \quad \frac{\partial u}{\partial t} + \mathrm{div}\,\vec{q} = 0 \; !$$

$$\rightarrow \quad \frac{\partial T}{\partial t} = \frac{1}{\rho c}\frac{\partial u}{\partial t} = \frac{\lambda}{\rho c}\Delta T$$

Wärmeleitungsgl.

$$\frac{\partial u}{\partial t} = -\operatorname{div}\vec{q} = +\lambda \operatorname{div}\operatorname{grad} T$$

$$\vec{q} = -\lambda \operatorname{grad} T$$

$$a = \lambda / \rho c$$

$\rightarrow$ mathematisches Problem:

finde Lösung $T(x,t)$ der partiellen DGL

$$\frac{\partial T}{\partial t} = a\,\frac{\partial^2 T}{\partial x^2}$$

zur Randbedingung $T(x,0) = T_0(x)$!

Klausur am Do, 31. Juli:

~~$8^{00} - 11^{00}$~~ $\longrightarrow$ $9^{00} - 12^{00}$ HS I + HS II

Lösung mittels Fouriertrafo in x:

$$T(x,t) \xrightarrow{\mathcal{F}} \hat{T}(k,t)$$

$$\frac{\partial^2 T}{\partial x^2}(x,t) \longrightarrow -k^2\,\hat{T}(k,t)$$

WLG: $\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \xrightarrow{\mathcal{F}} \boxed{\frac{\partial}{\partial t}\hat{T}(k,t) = -ak^2\,\hat{T}(k,t)}$

gewöhnliche DGL für $\hat{T}(k,t)$!

der Art: $\dot{y}(t) = -\alpha\, y(t)$

$\rightsquigarrow$ Lsg: $\quad Y(t) = Y_0 \, e^{-\alpha t}$

d.h. $\quad \boxed{\hat{T}(k,t) = \hat{T}_0(k)\, e^{-a k^2 t}} \quad (*)$

$$\hat{T}_0(k) = \mathcal{F}[T_0(x)]$$

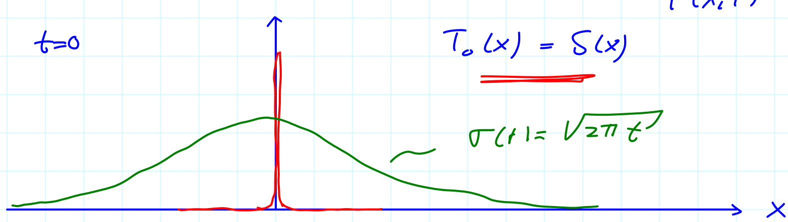
$\mathcal{F}^{-1}$

$$T(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{T}(k,t)\, e^{ikx}$$

$$\boxed{T(x,t) \underset{(*)}{=} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{T}_0(k)\, e^{-ak^2 t + ikx}}$$

Kurz: $T(x,0) = T_0(x) \xrightarrow{\mathcal{F}} \hat{T}_0(k) \xrightarrow{t} e^{-ak^2 t}\hat{T}_0(k) = \hat{T}(k,t) \xrightarrow{\mathcal{F}^{-1}} T(x,t)$

Bsp:



$$T_0(x) = \delta(x) \xrightarrow{\mathcal{F}} \hat{T}_0(k) = \frac{1}{\sqrt{2\pi}} \xrightarrow{t} \hat{T}(k,t) = \frac{e^{-ak^2 t}}{\sqrt{2\pi}}$$

$$\xrightarrow{\mathcal{F}^{-1}} T(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \frac{e^{-atk^2 + ixk}}{\sqrt{2\pi}} = \frac{1}{2\pi}\sqrt{\frac{\pi}{at}}\, e^{-x^2/4at}$$

Gauß-Int.

$$= \frac{1}{\sqrt{4\pi a t}}\, e^{-\frac{x^2}{4at}}, \quad \text{mit } \sigma(t) = \sqrt{2at}$$

$$\rightsquigarrow \boxed{T(x,t) = \frac{1}{\sqrt{2\pi\,\sigma(t)^2}}\, e^{-x^2/2\sigma^2(t)}} =: T(x,t)$$

Gauß-Funkt. mit zeitabhängiger Varianz $\sigma(t) = \sqrt{2at}$

Lösung zum A.W. $T_0(x) = \delta(x)$

↳ alt. Lösungsverfahren:

$$T(x,0) = \int dy\, T_0(y)\, \delta(x-y) \qquad \checkmark$$

$$\Big\downarrow t \qquad\qquad\qquad\qquad \Big\downarrow t$$

$$T(x,t) = \int_{-\infty}^{\infty} dy\, T_0(y)\, T(x-y,t)$$

$$= \int_{-\infty}^{\infty} dy\, T_0(y)\, \frac{1}{\sqrt{2\pi\sigma_t}}\, e^{-(x-y)^2/2\sigma_t^2}$$

Def.: Faltung

Die Faltung zweier Funktionen f und g ist die Funktion $f*g$ def. durch

$$(f*g)(x) := \int_{-\infty}^{\infty} dy\, f(y)\, g(x-y)$$

↳ Faltungssatz:

$$\mathcal{F}[f*g] = \sqrt{2\pi}\, \mathcal{F}[f] \cdot \mathcal{F}[g]$$

$$\widehat{f*g}(k) = \sqrt{2\pi}\, \hat{f}(k) \cdot \hat{g}(k)$$

$$(f*g)(x) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \hat{f}(k)\, e^{iky} \int_{-\infty}^{\infty} \frac{d\tilde{k}}{\sqrt{2\pi}}\, \hat{g}(\tilde{k})\, e^{i\tilde{k}(x-y)}$$

$$= \int dk \int d\tilde{k}\, \hat{f}(k)\, \hat{g}(\tilde{k}) \underbrace{\left(\int_{-\infty}^{+\infty} \frac{dy}{2\pi}\, e^{iy(k-\tilde{k})} \right)}_{=\delta(k-\hat{k})} e^{i\tilde{k}x}$$

$$= \int \frac{dk}{\sqrt{2\pi}} \, \sqrt{2\pi}\, \hat{f}(k) \cdot \hat{g}(k) \, e^{ikx} \overset{!}{=} \int \frac{dk}{\sqrt{2\pi}} \, \widehat{(f * g)}(k) \, e^{ikx}$$

$$\text{i.e.} \quad \widehat{f * g}(k) = \sqrt{2\pi} \, \hat{f}(k) \, \hat{g}(k)$$