Wdhlg.:

- komplexe Fkt. $f: \underset{\substack{\cup \\ \mathbb{C}}}{G} \to \mathbb{C}$, $\quad z \mapsto f(z)$

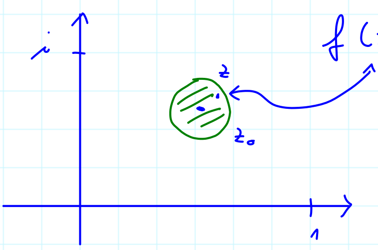
- komplexe Ableitung von f in z_0:

$$\boxed{\frac{df}{dz}(z_0) \equiv f'(z_0) = \lim_{h \to 0} \frac{1}{h}\big(f(z_0+h) - f(z_0)\big)}$$

sofern existent: $\rightsquigarrow$ f komplex diffbar in z_0

$\rightarrow$ lineare Näherung:

$$f(z) \approx f(z_0) + f'(z_0)(z_0 - z)$$



- $f: G \to \mathbb{C}$ holomorph $:\Leftrightarrow$ $\forall z_0 \in G$: f diffbar in z_0

Beispiele holomorpher Fkt.en:

- ganzrationale Fkten: z^4, z^{-4}, $\dfrac{z^2+3z}{5z-z^3}$, ...

- Fkt.en def. durch Potenzreihen:

$\exp(z)$, $\sin(z)$, $\cosh(z)$, ...

- f, g holomorph $\rightsquigarrow$ $f+g$, αf, $f \cdot g$, f/g, $f \circ g$, f^{-1} holomorph

$\rightarrow$ z.B. $\ln\left(\dfrac{e^z + \sin(z)}{3z^2 + \cos^2(z)}\right)$ holomorph

Ableitungsregeln:

- $(f+g)' = f' + g'$
- $(\alpha f)' = \alpha f'$
- $(fg)' = f'g + fg'$
- $(f/g)' = \dfrac{f'g - fg'}{g^2}$
- $(f \circ g)' = (f' \circ g)\, g'$
- $(f^{-1})' = \dfrac{1}{f' \circ f^{-1}}$

Beispiele nicht holomorpher Fkten:

z^*, $|z| = (z^* z)^{1/2}$, $\mathrm{Re}\, z = \frac{1}{2}(z + z^*)$, $\mathrm{Im}\, z = \frac{1}{2i}(z - z^*)$ ...

komplexe vs reelle Diff.barkeit:

$$f: G \to \mathbb{C} \qquad \underline{\text{komplex diff.bar}} \text{ in } z_0$$
$$z \mapsto f(z)$$

$$\Leftrightarrow \quad f: G \to \mathbb{R}^2 \qquad \underline{\text{reell diff.bar}} \text{ in } p_0$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix}$$

$z = x + iy = \begin{pmatrix} x \\ y \end{pmatrix}$

$f = u + iv = \begin{pmatrix} u \\ v \end{pmatrix}$

$z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = p_0$

<u>und</u>

$$df_{p_0} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \overset{!}{=} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad (*)$$

d.h. f reell diff.bar in p_0 und $u = \mathrm{Re} f$, $v = \mathrm{Im} f$ erfüllen die <u>Cauchy-Riemann-DGLen</u>

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

(*) :
$$\boxed{df_{p_0} = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \lambda R_\alpha}$$

$\lambda = (a^2 + b^2)^{1/2}$

$\tan\alpha = b/a$

isotrope Dehnung / Stauchung

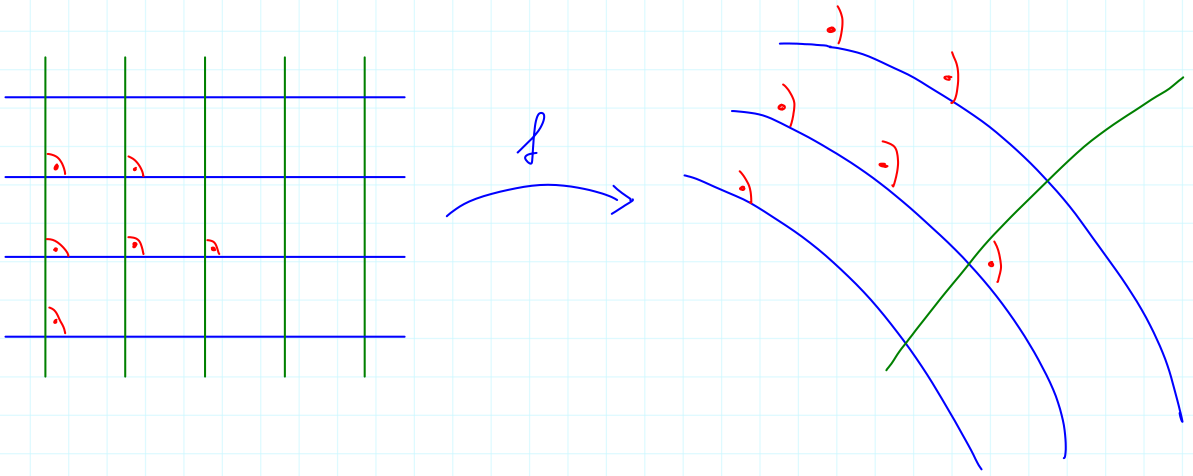
Drehung

$\leadsto$ **holomorphe Funktionen sind genau die <u>winkeltreuen</u> Abbildungen $\mathbb{C} \to \mathbb{C}$**

d.h. holomorphe Fkten = 2D <u>konforme</u> Abb.en

$\shortparallel$

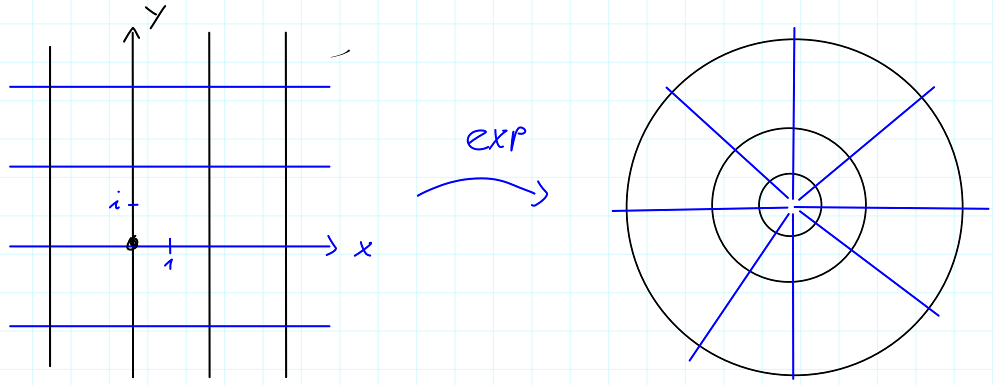
winkeltreu



Beispiel

$$\exp: \mathbb{C} \to \mathbb{C}$$

$$z = x+iy \longmapsto e^{z} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)$$



zu (*):

$$\underbrace{f'(z_0)}_{a+ib}\ \underbrace{h}_{x+iy} = (a+ib)(x+iy) = ax-by + i(bx+ay)$$

$$= \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\overset{!}{=} df_{z_0}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

kurz: $f'(z_0) \overset{!}{=} \lambda e^{i\varphi}$ $\rightsquigarrow$ $f'(z_0)\,h \overset{!}{=} \underline{\lambda\, e^{i\varphi}}\, h \overset{!}{=} \underline{df_p}\begin{pmatrix} x \\ y \end{pmatrix}$

$\varphi, \lambda \in \mathbb{R}$

Dehnung Stauchung

Drehung in $\mathbb{C} = \mathbb{R}^2$

Bemerkung: holomorphe Fkt. $f = u + iv$:

$\rightarrow$ $\boxed{\Delta u = 0 \quad , \quad \Delta v = 0}$

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

d.h. Real- und Imaginärteil einer holomorphen Fkt. sind harmonische Fkten.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial x}}_{=\frac{\partial v}{\partial y}}\Big) + \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial u}{\partial y}}_{=-\frac{\partial v}{\partial x}}\Big)$$

C.-R. DGL!

$$= \frac{\partial^2}{\partial x \partial y} v - \frac{\partial^2}{\partial y \partial x} v = 0 \quad \checkmark$$

S. v. Schwartz

$\Delta v \overset{!}{=} 0$ analog!

$\Rightarrow$ direkte <u>Anwendung in der Elektrostatik</u> (2D!)

im Vakuum $\Rightarrow$ $\rho = 0$

$\frac{\partial B}{\partial t} = 0$

E.S.: $\operatorname{div} \vec{E} = 0$, $\operatorname{rot} \vec{E} = 0$

$\Rightarrow$ $\vec{E} = -\operatorname{grad} \phi$

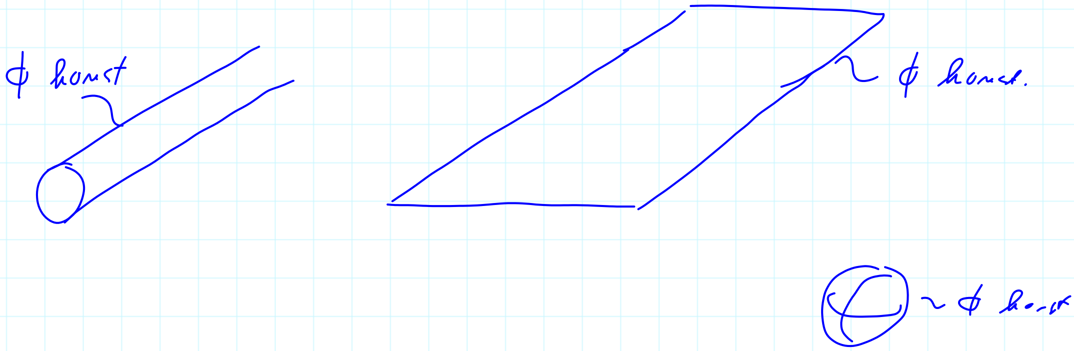
$\vec{\nabla} \cdot \vec{\nabla}$

$\rightarrow$ $\underbrace{\operatorname{div} \operatorname{grad}}\, \phi = 0$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

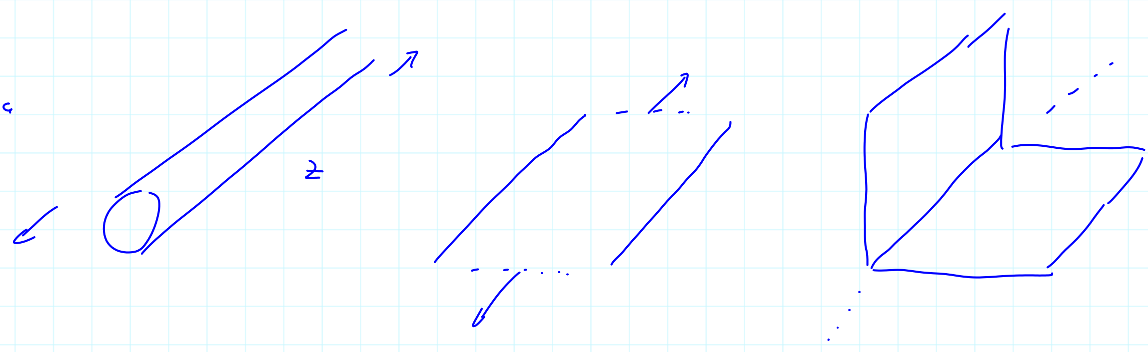
E.S. im Vakuum: $\boxed{\Delta \phi = 0}$ Laplace-Gl.

+ Randbedingungen!!



hier: Randbed. so, dass $\phi(x,y,\cancel{z}) \stackrel{!}{=} \phi(x,y)$ ...

etwa

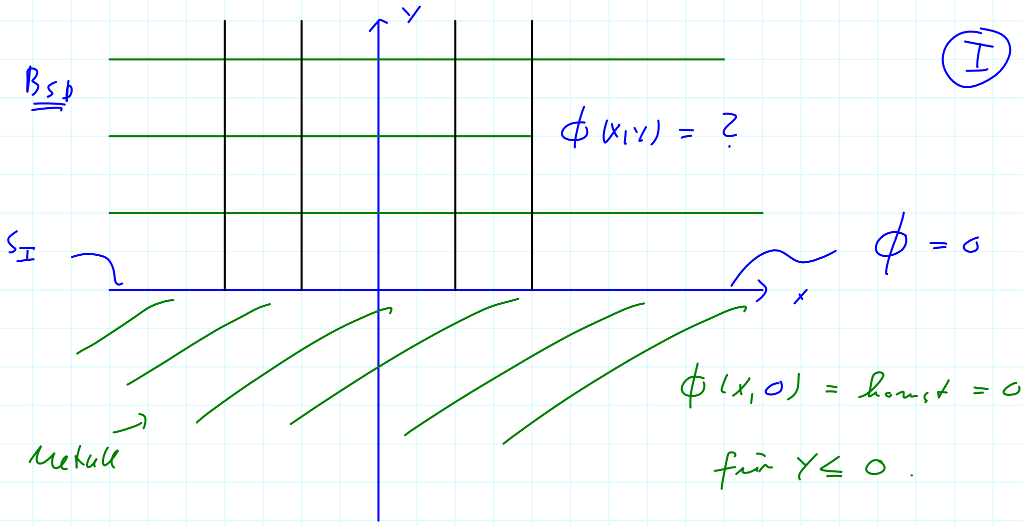


$\rightarrow$ setze $\phi(x,y) = \begin{cases} \mathrm{Re}\, f(x+iy) \\ \mathrm{Im}\, f(x+iy) \end{cases}$,

wobei f holomorph und $\phi(x,y)$ erfüllt Randbedingungen!

[$\leadsto \Delta \phi = 0$!]

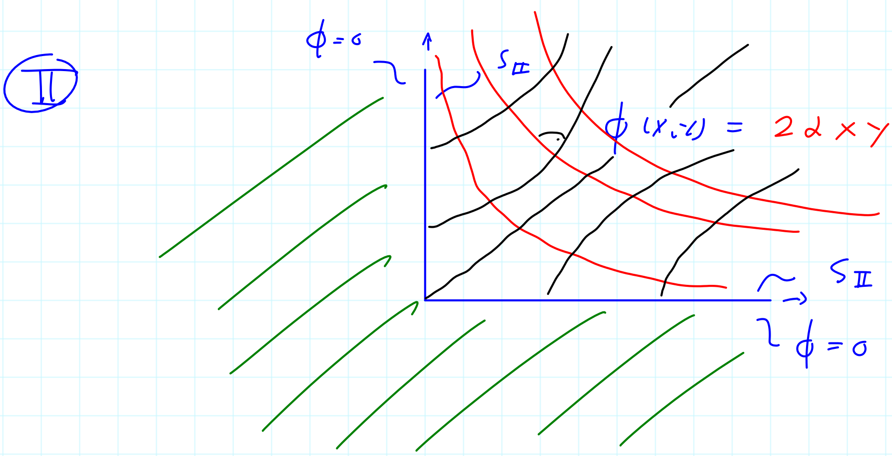
Bsp.



setze $\phi(x,y) = \operatorname{Im} f(x+iy) = \operatorname{Im} \alpha(x+iy) = \alpha y$ ✓

$f(z) = z\alpha$

$\phi(x,0) = 0$ ✓



Idee: wähle holomorphe Fkt g so,

dass $g(S_{II}) \overset{!}{=} S_I$

$\to$ $f \circ g$ holomorph und $f \circ g|_{S_{II}} = 0$!

$\to$ $\phi_{II}(x,y) \overset{!}{=} \operatorname{Im} f \circ g$

setze $g = z^2$! ✓

$\phi_{II}(x,y) = \operatorname{Im} \underset{\alpha z}{f}((x+iy)^2) = \alpha \operatorname{Im}(x^2 - y^2 + 2ixy)$

$= 2\alpha xy$; $\phi = u$

$\to y = \frac{u}{2\alpha} \frac{1}{x}$