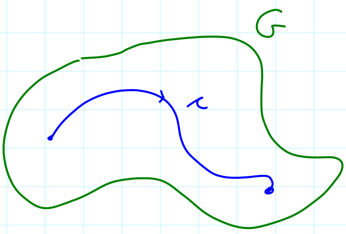
Komplexes Wegintegral

Fkt. $f : G \to \mathbb{C}$, Weg γ in G



$$\gamma : [t_1, t_2] \to G$$

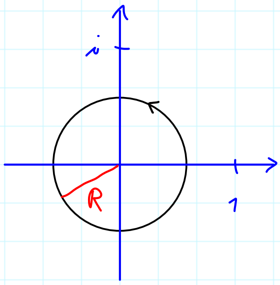
$$t \mapsto \gamma(t) = \gamma_1(t) + i\gamma_2(t)$$

$$\boxed{\int_\gamma f(z)\, dz := \int_{t_1}^{t_2} f(\gamma(t)) \cdot \gamma'(t)\, dt}$$

$$:= \int_{t_1}^{t_2} \mathrm{Re}\Big(f(\gamma(t))\, \gamma'(t)\Big)\, dt + i \int_{t_1}^{t_2} \mathrm{Im}\big(f(\gamma(t))\, \gamma'(t)\big)\, dt.$$

__Bsp:__ $f(z) = z$, Kreisweg um $z=0$, Radius R

$$K_R = \gamma : [0, 2\pi] \to \mathbb{C}$$
$$t \mapsto Re^{it} \quad \checkmark$$



$$\gamma'(t) = iRe^{it} = i\gamma(t)$$

$$\to \int_{K_R} f(z)\, dz \overset{f=z}{=} \int_0^{2\pi} \gamma(t)\gamma'(t)\, dt = i\int_0^{2\pi} \gamma(t)^2\, dt$$

$$= iR^2 \int_0^{2\pi} e^{i2t}\, dt =$$

$$= iR^2 \int_0^{2\pi} \big(\cos(2t) + i\sin(2t)\big)\, dt = 0$$

$$\int_{K_R} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\gamma(t)} \cdot \underbrace{\gamma'(t)}_{= i\gamma(t)} dt = i \int_0^{2\pi} \frac{\cancel{\gamma(t)}}{\cancel{\gamma(t)}}\, dt = 2\pi i$$

$n \in \mathbb{Z}$:

$$\gamma'(t) = i\gamma(t)$$

$$\int_{K_R} z^n\, dz = \int_0^{2\pi} \gamma(t)^n \gamma'(t)\, dt = i\int_0^{2\pi} \gamma(t)^{n+1}\, dt$$

$$= iR^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt = \begin{cases} 2\pi i & : n = -1 \\ 0 & : n \neq -1 \end{cases}$$

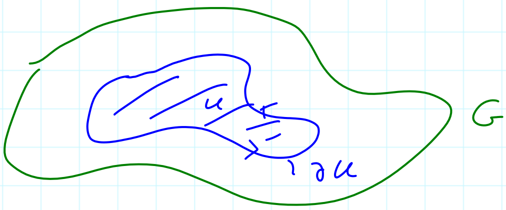
$$\boxed{\int_{K_R} z^n\, dz = 2\pi i\, \delta_{n,-1}}$$

Cauchyscher Integralsatz

$f : G \to \mathbb{C}$ holomorph, Gebiet $U \subset G$,

U einfach zusammenhängend

$$\int_{\partial U} f(z)\,dz = 0$$



$c = \partial U$

Beweisidee: "holomorphe Fkt." $\hat{=}$ "konservatives VF"

$f = u + iv$, $\quad c(t) = c_1(t) + i\,c_2(t)$, $\quad t \in [t_0, t_1]$

$$\rightarrow \int_c f(z)\,dz = \int_{t_0}^{t_1} (u + iv)\Big|_{z=c(t)} \cdot \left(c_1'(t) + i\,c_2'(t) \right) dt$$

$$= \int_{t_0}^{t_1} (u\,c_1' - v\,c_2') + i\,(v\,c_1' + u\,c_2')\,dt$$

$$\rightarrow \operatorname{Re} \int_c f(z)\,dz = \int_{t_0}^{t_1} \left\langle \begin{pmatrix} u \\ -v \end{pmatrix}, \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} \right\rangle dt = \int_c \vec{F}\,d\vec{l} = 0 \;!$$

mit $\vec{F} = \begin{pmatrix} u \\ -v \end{pmatrix}$

$\vec{F}$ konservativ: $\quad \dfrac{\partial F_x}{\partial y} = \dfrac{\partial u}{\partial y} \overset{\text{C. R. DGL}}{=} -\dfrac{\partial v}{\partial x} = \dfrac{\partial F_y}{\partial x}$

für $\operatorname{Im} \int_C f(z)\,dz$ analog !