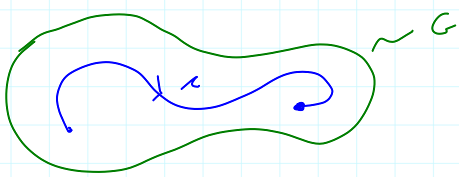
Wdhlg.: Komplexes Wegintegral:

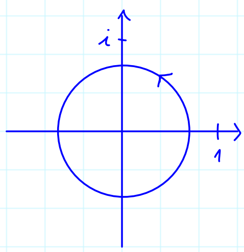


$f : G \to \mathbb{C}$

$\gamma : [t_1, t_2] \to G$

$$\boxed{\int_\gamma f(z)\,dz := \int_{t_1}^{t_2} f(\gamma(t))\,\gamma'(t)\,dt}$$

Bsp: Kreisweg $\gamma = K_R$: $\gamma(t) = R e^{it}$, $t \in [0, 2\pi[$



$$\rightsquigarrow \int_\gamma z^n\,dz = \begin{cases} 2\pi i & : n = -1 \\ 0 & : n \neq -1 \end{cases}$$

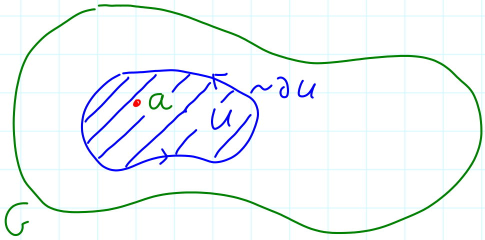
- Cauchyscher Integralsatz:

$f : G \to \mathbb{C}$ holomorph, $U \subset G$ einf. zusammenhängend

$$\rightsquigarrow \boxed{\int_{\partial U} f(z)\,dz = 0}$$

∂U : geschlossener Weg

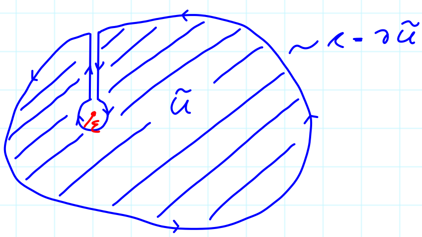
Cauchysche Integralformel



$$\boxed{f(a) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z-a}\,dz}$$

$f: G \to \mathbb{C}$ holomorph, $U \subset G$ einf. zshgnd, $\partial U \subset G$, $a \in U$

Beweis:



- $\left.\frac{f(z)}{z-a}\right| U \setminus \{a\}$ holomorph
- $\tilde{U} \subset U \setminus \{a\}$ einf. zshgnd.
- $\gamma = \partial \tilde{U}$

$$\rightsquigarrow \quad 0 \overset{!}{=} \int_\gamma \frac{f(z)}{z-a}\,dz = \int_{\partial U} \frac{f(z)}{z-a}\,dz - \int_{K_\varepsilon(a)} \frac{f(z)}{z-a}\,dz$$

d.h.
$$\int_{\partial U} \frac{f(z)}{z-a}\,dz = \int_{K_\varepsilon(a)} \frac{f(a) + f'(a)(z-a)}{z-a}\,dz + o(\varepsilon)$$

$$= f(a) \underbrace{\int_{K_\varepsilon(a)} \frac{dz}{z-a}}_{= \int_{K_\varepsilon(0)} \frac{1}{z}\,dz = 2\pi i} + \underbrace{\int_{K_\varepsilon(a)} f'(a)\,dz}_{= o(\varepsilon)} + o(\varepsilon)$$

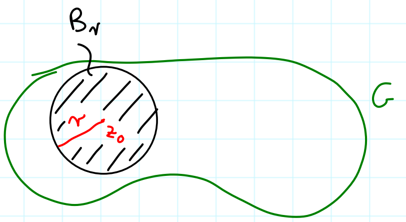
$\varepsilon \to 0$

$$\rightsquigarrow \quad f(a) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(z)}{z-a}\,dz \quad \square$$

2) Potenzreihenentwicklungssatz

Eine holomorphe Fkt. $f: G \to \mathbb{C}$ kann um $z_0 \in G$ auf Kreisscheibe $B_r \subset G$ um z_0 durch Potenzreihe entwickelt werden

$$f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \qquad (z \in B_r),$$



mit eindeutig bestimmten Koeffizienten

$$b_n = \frac{1}{2\pi i} \int_{K_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

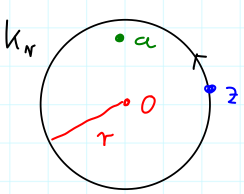
Folgerung:

Jede holomorphe Fkt. ist beliebig oft differenzierbar!

O. B. d. A. $z_0 = 0$:

Cauch. Integralformel

$$f(a) = \frac{1}{2\pi i} \int_{K_r} \frac{f(z)}{z-a}\, dz$$



$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \int_{K_r} \frac{f(z)}{z^{n+1}}\, dz \right)}_{b_n} a^n$$

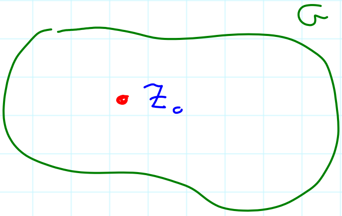
$$\frac{1}{z-a} = \frac{1}{z} \, \frac{1}{1-a/z} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$|a/z| < 1$

Singularitäten, meromorphe Fkt., Laurent-Reihe

Residuensatz

$f : G \setminus \{z_0\} \longrightarrow \mathbb{C}$ holomorph



Singularität von f in z_0 :

hebbar : $\exists\, a \in \mathbb{C} : \quad f(z) = \begin{cases} f(z) & : z \neq z_0 \\ a & : z = z_0 \end{cases}$ holomorph

Pol von Ordnung k : ($k \in \mathbb{N}$)

- z_0 hebbare Sing. von $(z-z_0)^k f(z)$
- z_0 nicht hebbar. Sing. v. $(z-z_0)^{k-1} f(z)$

wesentlich : weder hebbar noch Pol

⌜ $z \mapsto e^{1/z}$ $\rightarrow$ wesentliche Sing. bei $z = 0$ ⌟

Meromorphe Funktion

$f : G \to \mathbb{C}$ meromorph $:\Longleftrightarrow$ $f : G \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ holomorph

Polstellen von f

Laurent-Reihe einer meromorphen Funktion f um Pol z_0 von Ordnung k :

$$\underbrace{(z-z_0)^k f(z)}_{\text{holomorph!}} = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^{n-k}$$

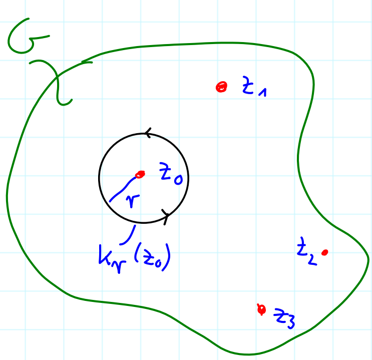
d.h.

$$\boxed{\begin{aligned} f(z) &= \sum_{n=-k}^{\infty} a_n (z-z_0)^n \\ &= \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1 (z-z_0) + \cdots \end{aligned}}$$

Laurent-Reihe von f um Pol z_0 von Ordnung k

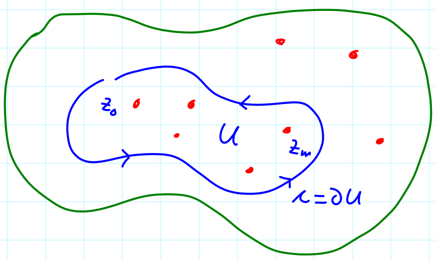
$\Rightarrow$ Residuum ("Rest") von f bei Pol z_0:

$$\boxed{\operatorname{Res}(f, z_0) := \frac{1}{2\pi i} \int_{K_r(z_0)} f(z)\, dz \overset{!}{=} a_{-1}}$$

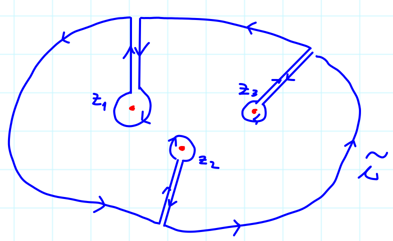


2> Residuensatz

$f : G \to \mathbb{C}$ meromorph mit Polstellen $z_1, \dots, z_m$ in $U \subset G$



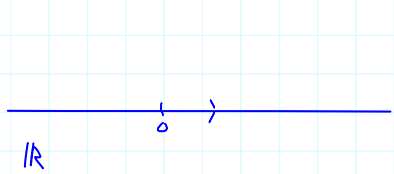
⇝ $$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_{n=1}^{m} \mathrm{Res}(f, z_n)}$$

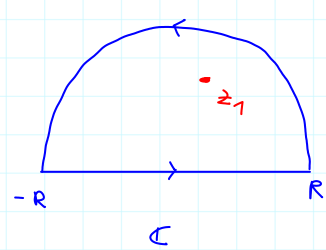


$$0 \overset{!}{=} \int_{\tilde{\gamma}} f(z)\,dz = \int_\gamma f(z)\,dz + \sum_{n=1}^{3} \underbrace{\int_{-K_r(z_n)} f(z)\,dz}_{= -2\pi i \,\mathrm{Res}(f, z_n)}$$

2> ermöglicht einfache Berechnung von (reellen!) Integralen:

etwa:
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_{\gamma_R} f(z)\,dz = 2\pi i \,\mathrm{Res}(f, z_1)$$





Methoden zur Residuums-Berechnung

1) mittels Laurent-Reihe:

$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \dots + \frac{c_{-1}}{z-z_0} + c_0 + c_1 (z-z_0) + \dots$$

$$\rightarrow \boxed{\operatorname{Res}(f, z_0) = c_{-1}}$$ ✓

2) z_0 Pol 1. Ordnung:

1) $\rightarrow$ $$\boxed{\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z)}$$ ✓

3) g, h holomorph, z_0 einfache Nullstelle von h

$\rightarrow$ 2) $$\boxed{\operatorname{Res}\left(\frac{g}{h}, z_0\right) = \frac{g(z_0)}{h'(z_0)}}$$ ✓

4) allg. Verfahren für z_0 Pol von Ordnung k:

$$\rightarrow (z-z_0)^k f(z) = c_{-k} + (z-z_0) c_{-k} + \dots + (z-z_0)^{k-1} c_{-1} + \dots$$

$$\rightarrow \boxed{\operatorname{Res}(f, z_0) = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((z-z_0)^k f(z) \right) \Big|_{z=z_0}}$$

Beispiele:

- $\operatorname{Res}\left(\frac{g(z)}{z-z_0}, z_0\right) \overset{2)}{=} g(z_0)$

- $\operatorname{Res}\left(\frac{g(z)}{\sin(z)}, 0\right) \overset{3)}{=} \frac{g(0)}{\cos(0)} = g(0)$

- $\operatorname{Res}\left(\frac{g(z)}{\sin(z)}, \pi\right) \overset{3)}{=} \frac{g(\pi)}{\cos(\pi)} = -g(\pi)$

- $\operatorname{Res}\left(\frac{g(z)}{z^3}, 0\right) \overset{4)}{=} \frac{1}{2!} \frac{d^2}{d^2 z} \cancel{z^3} \frac{g(z)}{\cancel{z^3}} \Big|_{z=0} = \frac{g''(0)}{2}$

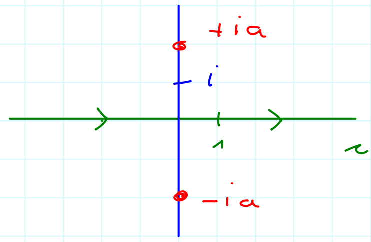
Integralbestimmung mittels Residuensatz:

<u>Beispiel</u>

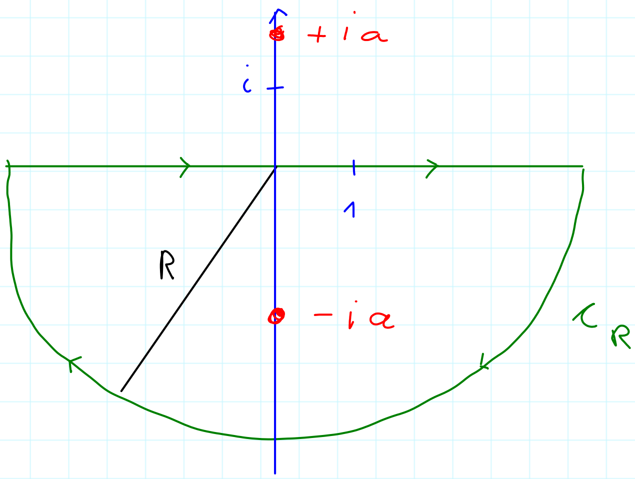
$$I(k,a) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\underbrace{x^2+a^2}_{=(x+ia)(x-ia)}}\, dx \,, \qquad k\in\mathbb{R},\ a\in\mathbb{R}_+$$

$\stackrel{!}{=}$ komplex. Wegintegral längs Weg "$\mathcal{C} = \mathbb{R}$":

$$\to \quad I(k,a) = \int_{\mathbb{R}} \underbrace{\frac{e^{-ikz}}{(z+ia)(z-ia)}}_{\substack{\text{meromorphe Fkt.}\\ \text{mit Polen 1. Ordnung bei } \mp ia}} dz$$



Für $k > 0$:



$$\rightsquigarrow \quad I(k,a) \stackrel{!}{=} \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{e^{-ikz}}{(z-ia)(z+ia)}\, dz = -2\pi i \underbrace{\mathrm{Res}(\dots, -ia)}_{\overset{2)}{=} -\frac{e^{-ka}}{2ai}}$$

$$= \frac{\pi}{a} e^{-ka}$$

für $k < 0$: schließe Weg in <u>obere</u> Halbebene

$$\to \quad I(k) = \frac{\pi}{a} e^{+ka}$$

d.h.

$$\boxed{I(k,a) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx = \frac{\pi}{a} e^{-|k|a}}$$