Partielle Integration

$$\boxed{\int_a^b f'(x)\,g(x)\,dx \overset{!}{=} f(x)\,g(x)\Big|_a^b - \int_a^b f(x)\,g'(x)\,dx}$$

$$\Big[\text{ denn } \int_a^b f' g\,dx = \int_a^b \{(f\cdot g)' - f g'\}\,dx = fg\Big|_a^b - \int_a^b f g'\,dx \Big]$$

Beispiele

1) $$\int_a^b x\,e^x\,dx \overset{\text{P.I.}}{=} x\,e^x\Big|_a^b - \int_a^b e^x\,dx = (x-1)\,e^x\Big|_a^b$$

2) $$\int_a^b x\ln x\,dx \overset{\text{P.I.}}{=} \frac{x^2}{2}\ln x\Big|_a^b - \underbrace{\int_a^b \frac{x^2}{2}\,\frac{1}{x}\,dx}_{=\,\frac{x^2}{4}\Big|_a^b} = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)\Big|_a^b$$

Substitution

$$\boxed{\int_a^b f(x)\,dx \overset{!}{=} \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\,g'(y)\,dy}$$

wobei $g : [g^{-1}(a), g^{-1}(b)] \to [a,b]$

<u>beliebige</u> umkehrbare, diff.bare Fkt

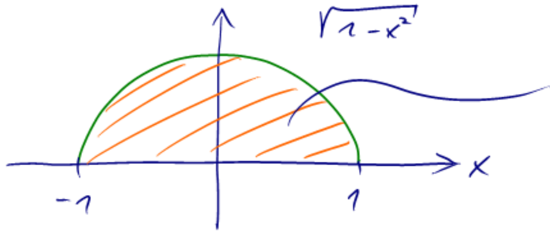
⌈ denn

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(g(g^{-1}(b))) - F(g(g^{-1}(a)))$$

$F' = f$

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} (F(g(y)))'\,dy = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\,g'(y)\,dy$$ ⌋

Beispiel: Flächeninhalt einer Kreisscheibe:



$$A = \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$A = \int_{-1}^{1} \sqrt{1-x^2}\, dx = \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\varphi}\, \cos\varphi\, d\varphi = \int_{-\pi/2}^{\pi/2} \cos^2\varphi\, d\varphi$$

Substit.: $x = \sin\varphi$, $\varphi \in [-\pi/2, +\pi/2]$

"Halbwinkelformel": $\boxed{\cos^2 x} = \frac{1}{2}(\cos^2 x + \cos^2 x) = \frac{1}{2}(1 + \underbrace{\cos^2 x - \sin^2 x}_{= \cos(2x)}) = \boxed{\frac{1}{2}(1+\cos(2x))}$

$$\rightarrow A = \int_{-\pi/2}^{+\pi/2} \cos^2\varphi\, d\varphi = \frac{1}{2} \underbrace{\int_{-\pi/2}^{\pi/2} 1\, d\varphi}_{= \pi} + \frac{1}{2} \underbrace{\int_{-\pi/2}^{\pi/2} \cos(2\varphi)\, d\varphi}_{= \frac{1}{2}\sin(2\varphi)\big|_{-\pi/2}^{\pi/2} = 0} = \frac{1}{2}\pi \quad \checkmark$$

Integration mittels Parameterableitung:

$$\boxed{\int_a^b \left(\frac{d}{d\lambda} f_\lambda(x)\right) dx \overset{!}{=} \frac{d}{d\lambda}\left(\int_a^b f_\lambda(x)\, dx\right)}$$

Beispiel:

$$\int_a^b x e^x\, dx = \int_a^b \left(\frac{d}{d\lambda} e^{\lambda x}\Big|_{\lambda=1}\right) dx = \frac{d}{d\lambda} \int_a^b e^{\lambda x}\, dx \Big|_{\lambda=1}$$

$$= \underbrace{\frac{d}{d\lambda} \frac{e^{\lambda x}}{\lambda}\Big|_{\lambda=1}}_{= x e^x - e^x} \Big|_a^b = (x-1)e^x \Big|_a^b$$

Ableitung eines Integrals nach der oberen Grenze:

$$\boxed{\frac{d}{dx} \int_a^{b(x)} f(y)\,dy = f(b(x)) \cdot b'(x)}$$

denn

$$\frac{d}{dx} \int_a^{b(x)} f(y)\,dy \underset{\substack{\uparrow \\ F' = f}}{=} \frac{d}{dx}\big(F(b(x)) - F(a)\big) = F'(b(x))\, b'(x) = f(b(x))\, b'(x)$$

insbesondere:

$(b(x) = x)$

$$\frac{d}{dx} \int_a^{x} f(y)\,dy = f(x)$$

$\longrightarrow$ $\boxed{F(x) := \int_a^x f(y)\,dy \quad \text{ist Stammfunktion von } f}$

Uneigentliche Integrale:

$$\boxed{\begin{aligned}\int_a^\infty f(x)\,dx &:= \lim_{b\to\infty}\int_a^b f(x)\,dx\\ \int_{-\infty}^b f(x)\,dx &:= \lim_{a\to-\infty}\int_a^b f(x)\,dx\end{aligned}}$$

für $f:]x_0, b] \to \mathbb{R}$: $\boxed{\int_{x_0}^b f(x)\,dx := \lim_{a\searrow x_0}\int_a^b f(x)\,dx}$

Beispiele:

- $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = 1$$
- $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\searrow 0}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\searrow 0} 2\sqrt{x}\Big|_a^1 = 2$$