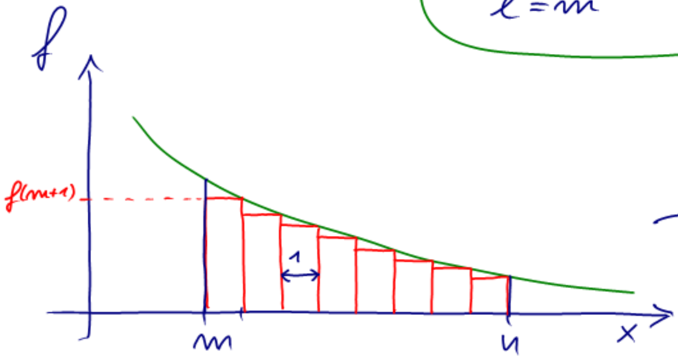
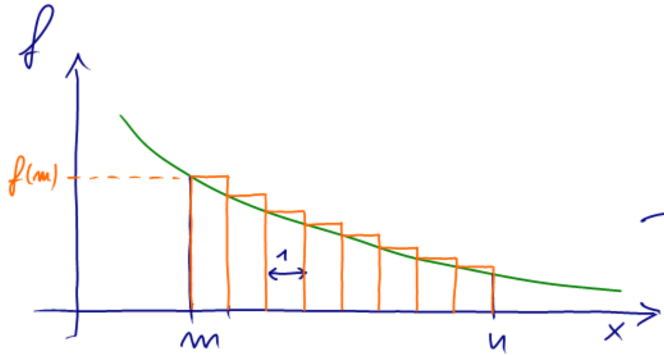
Abschätzung von Summen durch Integrale

$f : \mathbb{R} \to \mathbb{R}^+$ monoton fallende Funktion

$$\sum_{l=m}^{n} f(l) = ?$$ $\left(m, n \in \mathbb{N},\ m < n \right)$



$$\longrightarrow \quad \boxed{\sum_{l=m+1}^{n} f(l) \leq \int_m^n f(x)\,dx}$$



$$\longrightarrow \quad \boxed{\sum_{l=m}^{n-1} f(l) \geq \int_m^n f(x)\,dx}$$

$$\longrightarrow \quad \boxed{\int_m^{n+1} f(x)\,dx \le \sum_{\ell=m}^{n} f(x) \le \int_{m-1}^{n} f(x)\,dx}$$

$$\overset{2>}{\lim_{n\to\infty}} \quad \boxed{\int_m^{\infty} f(x)\,dx \le \sum_{\ell=m}^{\infty} f(x) \le \int_{m-1}^{\infty} f(x)\,dx}$$

Beispiele:

1) Wie groß ist $\sum_{\ell=m}^{n} \frac{1}{\ell}$?

mit obigen Abschätzungen:

$$\underbrace{\int_m^{n-1} \frac{1}{x}\,dx}_{\substack{\| \\ \ln x \Big|_m^{n-1} = \ln \frac{n-1}{m}}} \le \sum_{\ell=m}^{n} \frac{1}{\ell} \le \underbrace{\int_{m-1}^{n} \frac{1}{x}\,dx}_{\substack{\| \\ \ln x \Big|_{m-1}^{n} = \ln \frac{n}{m-1}}}$$

$$\rightarrow \quad \ln\frac{n-1}{m} \le \sum_{\ell=m}^{n} \frac{1}{\ell} \le \ln\frac{n}{m-1}$$

falls $n > m \gg 1$: $\qquad \ln(1-x) \simeq -x$

- $\ln\frac{n-1}{m} = \ln\left(\frac{n}{m}\left(1-\frac{1}{n}\right)\right) = \ln\frac{n}{m} + \ln\left(1-\frac{1}{n}\right) \simeq \ln\frac{n}{m} - \frac{1}{n}$
- $\ln\frac{n}{m-1} = -\ln\frac{m-1}{n} \simeq \ln\frac{n}{m} + \frac{1}{m}$

$$\rightarrow \quad \ln\frac{n}{m} - \frac{1}{n} \le \sum_{\ell=m}^{n} \frac{1}{\ell} \lesssim \ln\frac{n}{m} + \frac{1}{m}$$

insbes.: $\displaystyle\sum_{\ell=m}^{\infty} \frac{1}{\ell} \ge \int_m^{\infty} \frac{1}{x}\,dx = \infty$

d.h. $\displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ ist divergent!

2) Wie groß ist $\sum_{\ell=m}^{\infty} \frac{1}{\ell^2}$? $m > 1$

mit obigen Abschätzungen:

$$\frac{1}{m} = \int_m^{\infty} \frac{1}{x^2}\,dx \leq \sum_{\ell=m}^{\infty} \frac{1}{\ell^2} \leq \int_{m-1}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{m-1}$$

$\rightarrow$ insbes.: $\sum_{\ell=2}^{\infty} \frac{1}{\ell^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \leq 1$, konvergent!