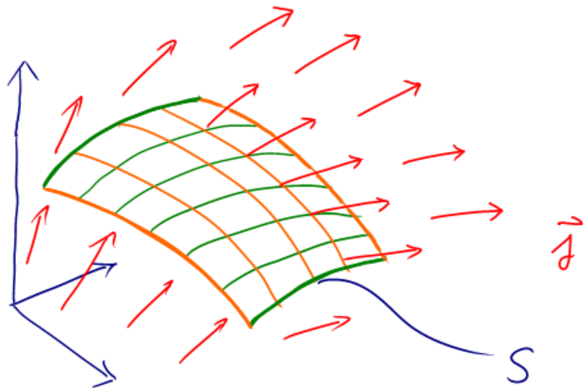
Flächen und Flächenintegrale

Motivation:

- Flächenstück S
- Massenstromdichte $\vec{j}$

$\Big\}\rightarrow$ Massenstrom I durch S:

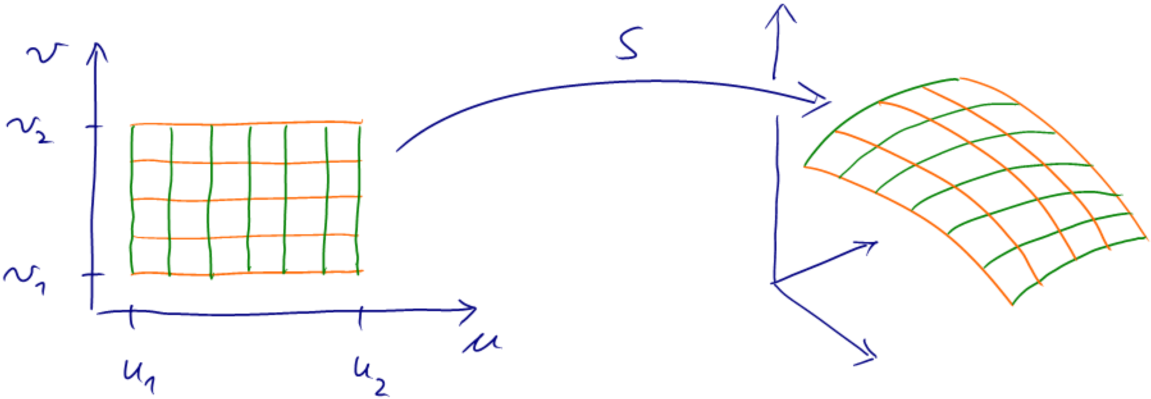
$$I = \int_S \vec{j}\, d\vec{f} = ?$$



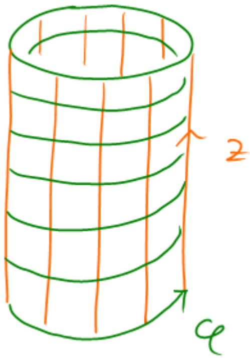
- Parametrisierung eines Flächenstücks S :

$$\text{Abb. } S : [u_1, u_2] \times [v_1, v_2] \longrightarrow \mathbb{R}^3$$

$$(u, v) \longmapsto \vec{S}(u, v)$$



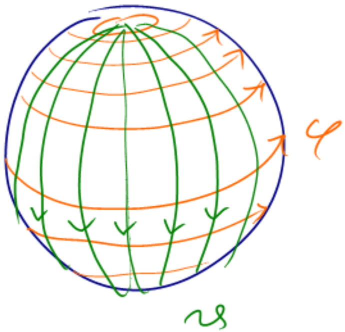
Beispiele: 1) Zylindermantel der Höhe H, Radius R:



$$S : [0, 2\pi] \times [0, H] \longrightarrow \mathbb{R}^3$$

$$(\varphi, z) \longmapsto \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

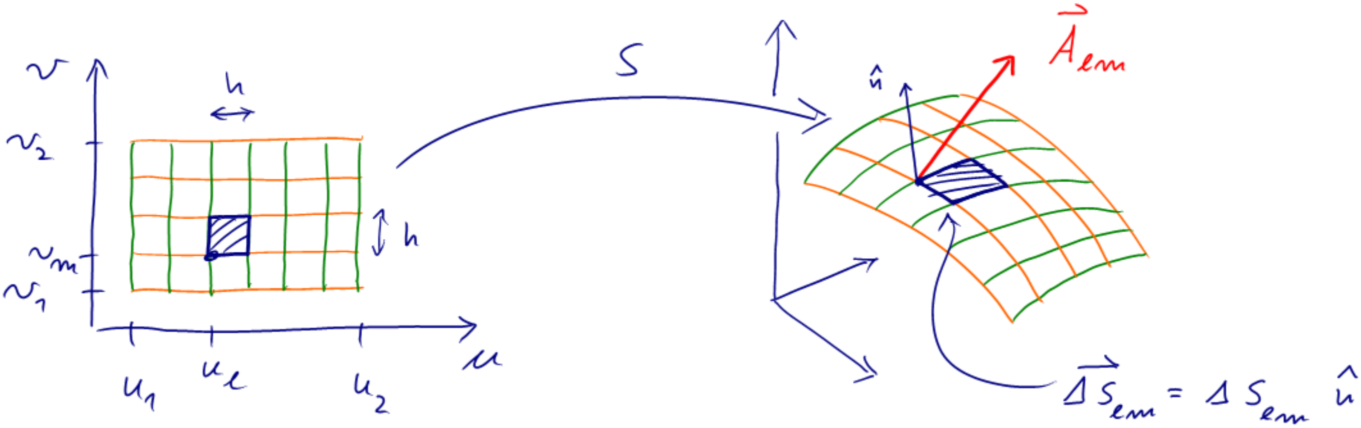
2) Sphäre



$$S : [0, \pi] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$$

$$(\vartheta, \varphi) \longmapsto R \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

Flächenintegral:



Fluss von $\vec{A}$ durch ΔS :

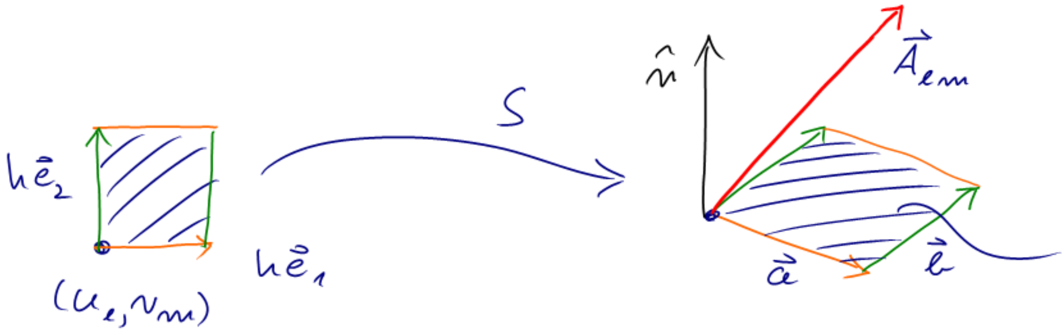
$$\boxed{I_{lm} = \langle \vec{A}_{lm}, \vec{\Delta S}_{lm} \rangle}$$

→ Fluss von $\vec{A}$ durch Flächenstück S :

$$\boxed{\int_S \vec{A}\, \vec{df} := \sum_{l,m} I_{lm}}$$

(im Limes $h \to 0$)

Fluss von $\vec{A}$ durch ΔS : $\quad I_{\ell m} = \langle \vec{A}_{\ell m}, \Delta\vec{s} \rangle$



$$\Delta\vec{S} = \vec{a} \times \vec{b}$$

mit $\boxed{\vec{a} = \frac{\partial \vec{S}}{\partial u}(u_\ell, v_m) \cdot h}$ und $\boxed{\vec{b} = \frac{\partial \vec{S}}{\partial v}(u_\ell, v_m)\, h}$:

$$I_{\ell m} = \left\langle \vec{A}(\vec{S}(u_\ell, v_m)), \frac{\partial \vec{S}}{\partial u}(u_\ell, v_m) \times \frac{\partial \vec{S}}{\partial v}(u_\ell, v_m) \right\rangle h^2$$

$$\rightarrow \int_S \vec{A}\, d\vec{f} = \sum_{\ell, m} \left\langle \vec{A}(\vec{S}(u_\ell, v_m)), \frac{\partial \vec{S}}{\partial u}(u_\ell, v_m) \times \frac{\partial \vec{S}}{\partial v}(u_\ell, v_m) \right\rangle h^2$$

$\rightarrow$ Flächenintegral vom Vf. $\vec{A}$ über Fläche S

$\equiv$ Fluss von $\vec{A}$ durch S :

$$\boxed{\int_S \vec{A}\, d\vec{f} := \int_{u_1}^{u_2} \int_{v_1}^{v_2} \left\langle \vec{A}(\vec{s}(u,v)), \frac{\partial \vec{s}}{\partial u}(u,v) \times \frac{\partial \vec{s}}{\partial v}(u,v) \right\rangle dv\, du}$$

analog: Flächenintegral des Skalarfelds g über Fläche S :

$$\boxed{\int_S g\, |d\vec{f}| := \int_{u_1}^{u_2} \int_{v_1}^{v_2} g(\vec{s}(u,v)) \left| \frac{\partial \vec{s}}{\partial u}(u,v) \times \frac{\partial \vec{s}}{\partial v}(u,v) \right| dv\, du}$$

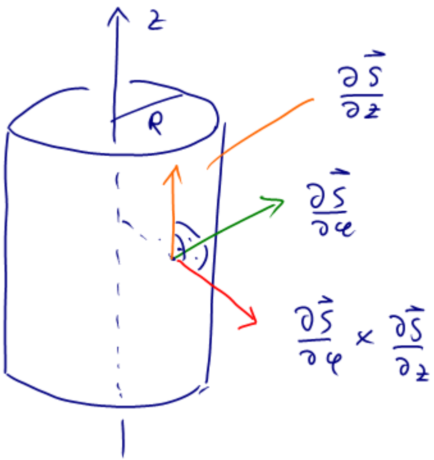
$g = 1$: Flächeninhalt von S : $\quad A(S) \equiv |S| := \int_S |d\vec{f}|$

Anwendung und Beispiele:

1) Integrale über Zylindermantel: $S: [0,2\pi]\times[0,H] \to \mathbb{R}^3$

$$(\varphi, z) \mapsto \vec{S}(\varphi,z) = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ z \end{pmatrix}$$

$$\to \quad \underline{\frac{\partial \vec{S}}{\partial \varphi}} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = \underline{R\,\vec{e}_\varphi}, \quad \underline{\frac{\partial \vec{S}}{\partial z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{\vec{e}_z}$$



$$\Rightarrow \quad \frac{\partial \vec{S}}{\partial \varphi} \times \frac{\partial \vec{S}}{\partial z} = R\,\vec{e}_\varphi \times \vec{e}_z = \underline{R\,\vec{e}_\varrho},$$

$$\left| \frac{\partial \vec{S}}{\partial \varphi} \times \frac{\partial \vec{S}}{\partial z} \right| = \underline{R}$$

S: Zylindermantel, Radius R, Höhe H

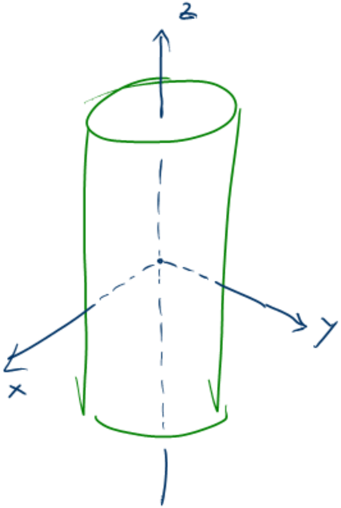
$$\to \quad \boxed{\begin{aligned} \int_S \vec{A}\, d\vec{f} &= \int_0^H \int_0^{2\pi} \langle \vec{A}(\varphi,z), \vec{e}_\varrho \rangle\, R\, d\varphi\, dz \\ \int_S g\, |d\vec{f}| &= \int_0^H \int_0^{2\pi} g(\varphi,z)\, R\, d\varphi\, dz \end{aligned}}$$

Beispiele:

a) Flächeninhalt des Zylindermantels:

$$A(S) = \int_S |d\vec{f}| = \int_0^H \int_0^{2\pi} R \, d\varphi \, dz = \int_0^H 2\pi R \, dz = \underline{\underline{2\pi R H}}$$

b) Hauptträgheitsmomente: Radius R, Höhe H, Masse M



$$\underline{\underline{I_{zz}}} = \int_S \lambda \, d^2 \, |d\vec{f}| = \frac{M R^2}{2\pi R H} \int_S |d\vec{f}| = \underline{\underline{M R^2}}$$

$$\lambda = \frac{M}{2\pi R H}$$

$$d_z(\varphi, z) = R$$

$$I_{xx} = \int_S \lambda \, d_x^2 \, |d\vec{f}| = I_{yy}$$

mit $\lambda = \frac{M}{2\pi R H}$, $d_x^2(\varphi, z) = z^2 + R^2 \sin^2\varphi$ also:

$$I_{xx} = \lambda \int_{-H/2}^{+H/2} \int_{0}^{2\pi} (z^2 + R^2 \sin^2\varphi)\, R\, d\varphi\, dz$$

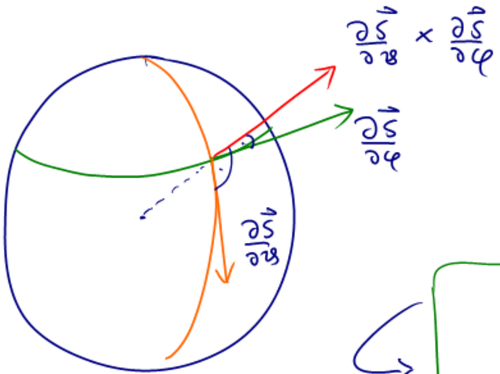
$$= \lambda \int_{-H/2}^{H/2} (2\pi z^2 + \pi R^2)\, R\, dz$$

$$= \frac{M}{2\pi R H} \left(\frac{\pi H^3}{6} + \pi R^2 H \right) R = M \left(\frac{H^2}{12} + \frac{R^2}{2} \right)$$

2) <u>Integrale über Sphäre</u> $S : [0,\pi]\times[0,2\pi] \longrightarrow \mathbb{R}^3$

$$(\vartheta, \varphi) \longmapsto \vec{S}(\vartheta,\varphi) = R\begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$$\rightarrow \quad \frac{\partial \vec{S}}{\partial \vartheta} = R\begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ -\sin\vartheta \end{pmatrix} = R\,\vec{e}_\vartheta \, , \quad \frac{\partial \vec{S}}{\partial \varphi} = R\begin{pmatrix} -\sin\varphi \sin\vartheta \\ \cos\varphi \sin\vartheta \\ 0 \end{pmatrix} = R\sin\vartheta\, \vec{e}_\varphi$$



$$\frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} = R^2 \sin\vartheta \; \vec{e}_r$$

$$\left| \frac{\partial \vec{S}}{\partial \vartheta} \times \frac{\partial \vec{S}}{\partial \varphi} \right| = R^2 \sin\vartheta$$

S: <u>Sphäre</u>, Radius R

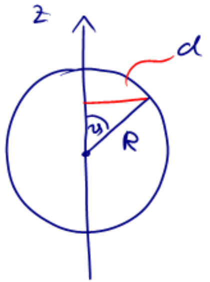
$$\boxed{\begin{aligned} \int_S \vec{A}\, d\vec{f} &= \int_0^{\pi}\int_0^{2\pi} \langle \vec{A}(\vartheta,\varphi), \vec{e}_r \rangle\, R^2 \sin\vartheta \, d\varphi\, d\vartheta \\ \int_S g\,|d\vec{f}| &= \int_0^{\pi}\int_0^{2\pi} g(\vartheta,\varphi)\, R^2 \sin\vartheta\, d\varphi\, d\vartheta \end{aligned}}$$

Beispiele:

a) Flächeninhalt der Sphäre:

$$A(S) = \int_S |d\vec{f}| = \int_0^{\pi}\int_0^{2\pi} R^2 \sin\vartheta \, d\varphi \, d\vartheta = \int_0^{\pi} 2\pi R^2 \sin\vartheta \, d\vartheta = \underline{\underline{4\pi R^2}}$$

b) Hauptträgheitsmoment einer Sphäre von Radius R und Masse M:



$$\underline{\underline{I_{zz}}} = \int_S \lambda d^2 |d\vec{f}| = \lambda \int_0^{\pi}\int_0^{2\pi} \underline{R^2 \sin^2\vartheta} \; \underline{\underline{R^2 \sin\vartheta \, d\varphi \, d\vartheta}}$$

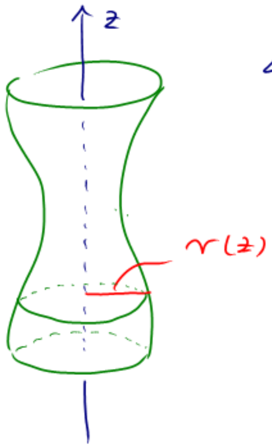
$d = \underline{R \sin\vartheta}$

$$= 2\pi\lambda R^4 \underbrace{\int_0^{\pi} \sin^3\vartheta \, d\vartheta}_{= 4/3} \underset{\lambda = M/4\pi R^2}{=} \underline{\underline{\frac{2}{3} M R^2}}$$

$\Big\lceil$ N.R.: $\displaystyle\int_0^{\pi} \sin^3\vartheta \, d\vartheta = \int_0^{\pi} \sin\vartheta \, d\vartheta - \int_0^{\pi} \sin\vartheta \cos^2\vartheta \, d\vartheta = -\cos\vartheta\Big|_0^{\pi} + \frac{\cos^3\vartheta}{3}\Big|_0^{\pi} = \frac{4}{3}$ $\Big\rfloor$

$\sin^2\vartheta = 1 - \cos^2\vartheta$

3) Integrale über Rotationsfläche:



$$S : [0, 2\pi] \times [z_1, z_2] \longrightarrow \mathbb{R}^3$$

$$(\varphi, z) \longmapsto \vec{S}(\varphi, z) = \begin{pmatrix} r(z)\cos\varphi \\ r(z)\sin\varphi \\ z \end{pmatrix}$$

$$\rightarrow \quad \frac{\partial \vec{S}}{\partial \varphi} \times \frac{\partial \vec{S}}{\partial z} = \begin{pmatrix} -r(z)\sin\varphi \\ r(z)\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} r'(z)\cos\varphi \\ r'(z)\sin\varphi \\ 1 \end{pmatrix} = \begin{pmatrix} r(z)\cos\varphi \\ r(z)\sin\varphi \\ -r(z)\,r'(z) \end{pmatrix}$$

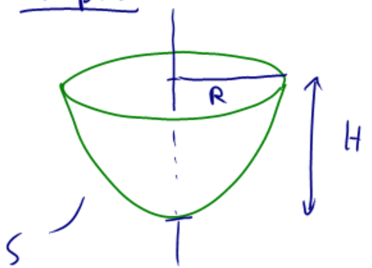
$$= r(z)\,\vec{e}_r - r(z)\,r'(z)\,\vec{e}_\varphi$$

$$\rightarrow \quad \left| \frac{\partial \vec{S}}{\partial \varphi} \times \frac{\partial \vec{S}}{\partial z} \right| = r(z)\sqrt{1 + r'(z)^2}$$

$$\rightarrow \quad \bullet \int_S \vec{A}\, d\vec{f} = \int_0^{2\pi} \int_{z_1}^{z_2} \langle \vec{A}(\varphi, z), \vec{e}_r - r'(z)\, \vec{e}_z \rangle\, r(z)\, dz\, d\varphi$$

$$\bullet \int_S g\, |d\vec{f}\,| = \int_0^{2\pi} \int_{z_1}^{z_2} g(\varphi, z)\, r(z) \sqrt{1 + r'(z)^2}\, dz\, d\varphi$$

Beispiel:



$$r(z) = \frac{R}{\sqrt{H}} \sqrt{z} \quad \text{für } z \in [0, H]$$

$$\Rightarrow r'(z) = \frac{R}{2\sqrt{H}} \frac{1}{\sqrt{z}}$$

$$\Rightarrow A(S) = \int_0^{2\pi} \int_0^{H} \frac{R}{\sqrt{H}} \sqrt{z} \sqrt{1 + \frac{R^2}{4H} \frac{1}{z}}\, dz\, d\varphi = 2\pi R \int_0^H \sqrt{\frac{z}{H} + \frac{R^2}{4H^2}}\, dz$$

$$= 2\pi R \cdot \frac{2H}{3} \left(\frac{z}{H} + \frac{R^2}{4H^2} \right)^{3/2} \Bigg|_0^H = \frac{4}{3} \pi R H \left(\left(1 + \frac{R^2}{4H^2}\right)^{3/2} - \left(\frac{R}{2H}\right)^3 \right)$$

$$\longrightarrow \quad A(S) = \frac{1}{6}\pi \frac{R^4}{H^2}\left(\left(1 + \frac{4H^2}{R^2}\right)^{3/2} - 1 \right)$$

$\underline{\underline{H << R}}$:

$$\left(1 + \frac{4H^2}{R^2}\right)^{3/2} - 1 \simeq \frac{6H^2}{R^2} \longrightarrow A(S) = \pi R^2$$

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