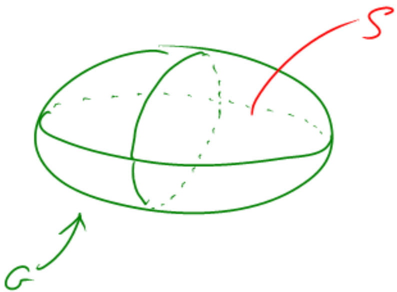
Volumengebiete und Volumenintegrale

Motivation: $\left.\begin{array}{l}\cdot \text{ Volumengebiet } G \\ \cdot \text{ Massendichte } \rho\end{array}\right\} \longrightarrow$ Masse in G:

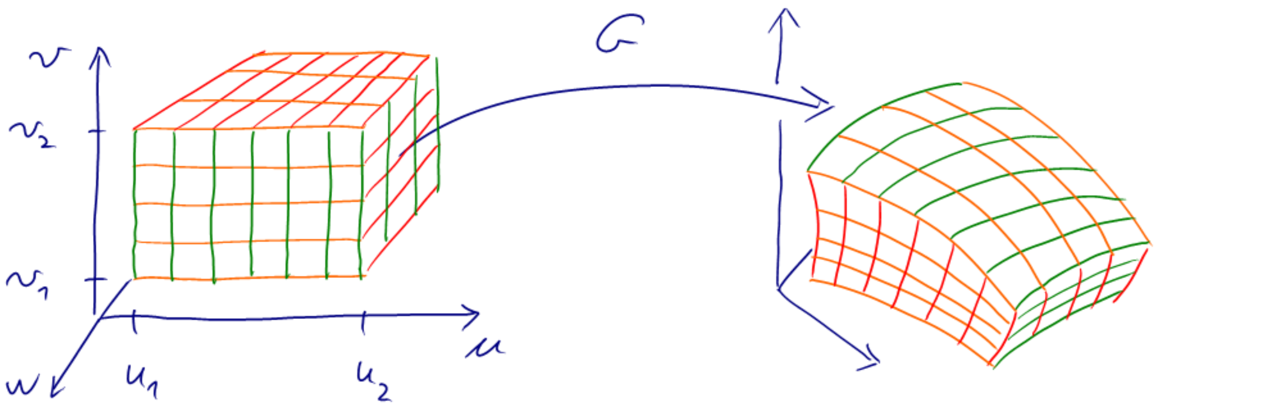
$$M_G = \int_G \rho \, dV = ?$$



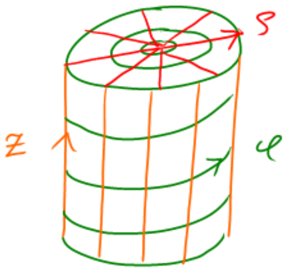
Parametrisierung eines Volumengebiets G:

Abb. $\quad G: [u_1, u_2] \times [v_1, v_2] \times [w_1, w_2] \longrightarrow \mathbb{R}^3$

$$(u, v, w) \longmapsto \vec{G}(u, v, w)$$



Beispiele: 1) <u>Zylinder</u> mit Radius R und Höhe H:

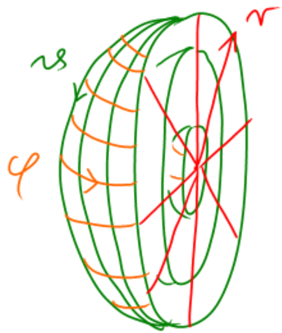


$$G : [0, 2\pi] \times [0, H] \times [0, R] \longrightarrow \mathbb{R}^3$$

$$(\varphi, z, \varrho) \longmapsto \vec{G}(\varphi, z, \varrho) = \begin{pmatrix} \varrho \cos\varphi \\ \varrho \sin\varphi \\ z \end{pmatrix}$$

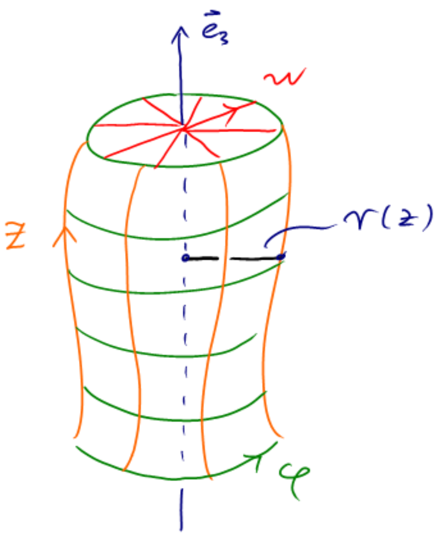
2) <u>Kugel</u> mit Radius : $G : [0, \pi] \times [0, 2\pi] \times [0, R] \longrightarrow \mathbb{R}^3$

$$(\vartheta, \varphi, r) \longmapsto \vec{G}(\vartheta, \varphi, r) = r \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$



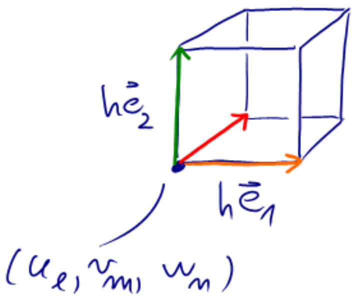
3) Rotationskörper : $G: [0,2\pi] \times [z_1, z_2] \times [0,1] \longrightarrow \mathbb{R}^3$

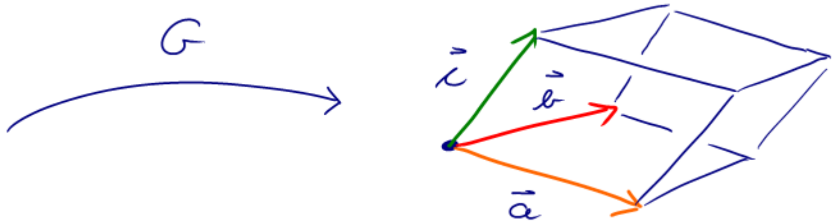
$$(\varphi, z, w) \longmapsto \vec{G}(\varphi, z, w) = \begin{pmatrix} w\, r(z) \cos\varphi \\ w\, r(z) \sin\varphi \\ z \end{pmatrix}$$



Volumenintegral

elementares Volumenelement:





$$\Delta G_{lmn} = \langle \vec{a} \times \vec{b}, \vec{c} \rangle$$

mit

$$\left.\begin{aligned} \vec{a} &= \frac{\partial \vec{G}}{\partial u} h \\ \vec{b} &= \frac{\partial \vec{G}}{\partial v} h \\ \vec{c} &= \frac{\partial \vec{G}}{\partial w} h \end{aligned}\right\} \longrightarrow \boxed{\Delta G_{lmn} = \left\langle \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}, \frac{\partial \vec{G}}{\partial w} \right\rangle h^3}$$

bewertet bei (u_l, v_l, w_m)

$\rightarrow$ Integral des Skalarfelds f über Volumengebiet G:

$$\int_G f \, dV := \lim_{h \to 0} \sum_{l,m,n} f(\vec{G}(u_l, v_m, w_n)) \left\langle \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}, \frac{\partial \vec{G}}{\partial w} \right\rangle h^3$$

d.h.
$$\boxed{\int_G f \, dV = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \int_{w_1}^{w_2} f(\vec{G}(u,v,w)) \left\langle \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}, \frac{\partial \vec{G}}{\partial w} \right\rangle_{(u,v,w)} dw \, dv \, du}$$

$f = 1$: Volumen des Gebiets G:

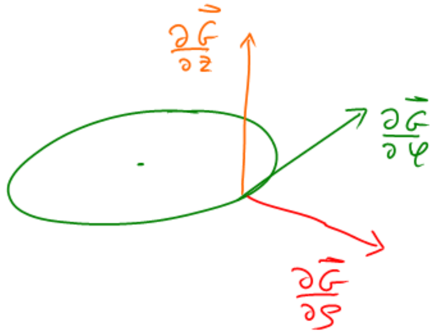
$$V(G) := \int_G dV$$

Anwendungen und Beispiele:

1) Integrale über Zylinder $G : [0, 2\pi] \times [0, H] \times [0, R] \to \mathbb{R}^3$

$$(\varphi, z, \rho) \mapsto \vec{G}(\varphi, z, \rho) = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix}$$

$$\rightsquigarrow \quad \frac{\partial \vec{G}}{\partial \varphi} = \rho\, \vec{e}_\varphi, \quad \frac{\partial \vec{G}}{\partial z} = \vec{e}_z, \quad \frac{\partial \vec{G}}{\partial \rho} = \vec{e}_\rho$$



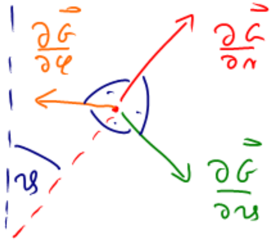
$$\Rightarrow \left\langle \frac{\partial \vec{G}}{\partial \varphi} \times \frac{\partial \vec{G}}{\partial z}, \frac{\partial \vec{G}}{\partial \rho} \right\rangle = \rho$$

$$\boxed{\int_G f \, dV = \int_0^{2\pi} \int_0^{H} \int_0^{R} f(\varphi, z, \rho)\, \underline{\underline{\rho \, d\rho \, dz \, d\varphi}}}$$

2) Integrale über Kugel $\quad G: [0,\pi]\times[0,2\pi]\times[0,R] \to \mathbb{R}^3$

$$(\vartheta,\varphi,r) \mapsto \vec{G}(\vartheta,\varphi,r) = \begin{pmatrix} r\cos\varphi\,\sin\vartheta \\ r\sin\varphi\,\sin\vartheta \\ r\cos\vartheta \end{pmatrix}$$

$$\rightarrow \quad \frac{\partial \vec{G}}{\partial \vartheta} = r\,\vec{e}_\vartheta \,, \quad \frac{\partial \vec{G}}{\partial \varphi} = r\sin\vartheta\,\vec{e}_\varphi \,, \quad \frac{\partial \vec{G}}{\partial r} = \vec{e}_r$$



$$\Rightarrow \quad \left\langle \frac{\partial \vec{G}}{\partial \vartheta} \times \frac{\partial \vec{G}}{\partial \varphi} , \frac{\partial \vec{G}}{\partial r} \right\rangle = \underline{\underline{r^2 \sin\vartheta}}$$

$$\Rightarrow \quad \boxed{\int_G f\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} f(\vartheta,\varphi,r)\;\underline{\underline{r^2\sin\vartheta\,dr\,d\varphi\,d\vartheta}}}$$

Beispiel: Volumeninhalt einer Kugel vom Radius R:

$$V = \int_G dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{R} r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

$$= 2\pi \, \frac{R^3}{3} \cdot 2 = \frac{4}{3} \pi R^3 \quad \checkmark$$