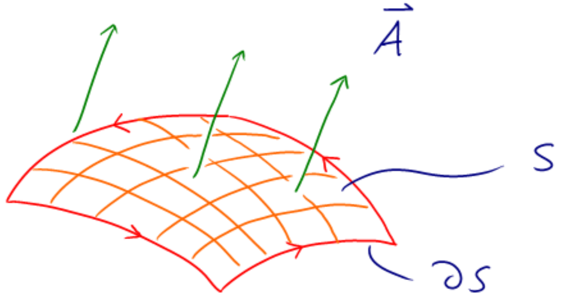
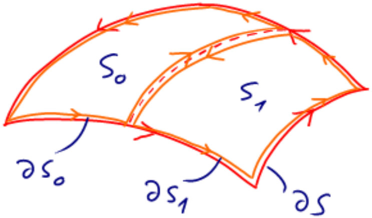
Satz von Stokes

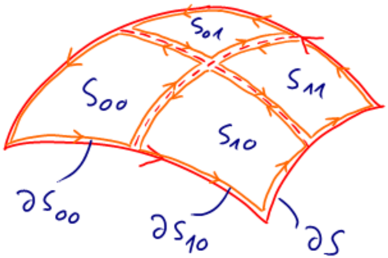


$$\boxed{\int_{\partial S} \vec{A}\, d\vec{l} = \int_{S} \mathrm{rot}\, \vec{A}\, d\vec{f}}$$

"Physikerbeweis" des Satz von Stokes:

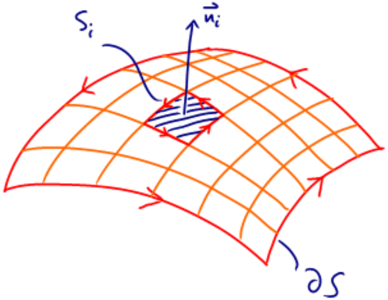


$$\int_{\partial S} \vec{A}\, d\vec{l} \overset{!}{=} \int_{\partial S_0} \vec{A}\, d\vec{l} + \int_{\partial S_1} \vec{A}\, d\vec{l}$$



$$= \int_{\partial S_{00}} \vec{A}\, d\vec{l} + \int_{\partial S_{01}} \vec{A}\, d\vec{l} + \int_{\partial S_{10}} \vec{A}\, d\vec{l} + \int_{\partial S_{11}} \vec{A}\, d\vec{l}$$

$$\vdots$$

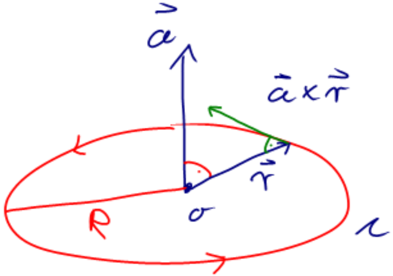


$$= \sum_{i=1}^{N} \underbrace{\frac{1}{|S_i|} \int_{\partial S_i} \vec{A}\, d\vec{l}}_{= \langle \vec{n}_i , \operatorname{rot} \vec{A} \rangle} |S_i| \overset{|S_i| \to 0}{=} \int_S \operatorname{rot} \vec{A}\, d\vec{f}$$

Beispiele und Anwendungen:

1) Vf. $\vec{A}(\vec{r}) = \vec{a} \times \vec{r}$ $\quad \rightarrow \quad \operatorname{rot} \vec{A} = 2\vec{a}$ (vgl. Vlsg. 27)

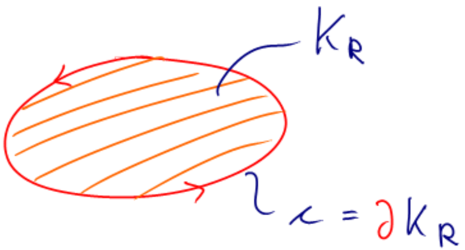
a)



C: Kreisweg $\perp \vec{a}$, Mittelpkt. O,
Radius R

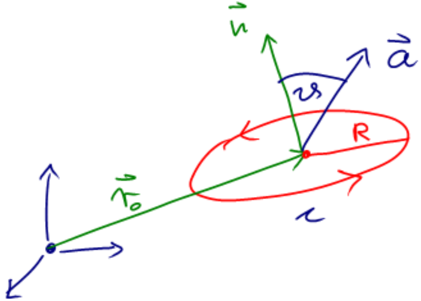
$$\Rightarrow \int_C \vec{A}\, d\vec{l} \underset{\substack{\uparrow \\ \vec{A} \parallel d\vec{l} \\ |\vec{A}| = R|\vec{a}|}}{=} 2\pi R\, R|\vec{a}| = 2\pi R^2 |\vec{a}| \,,$$

mit Satz von Stokes: $\quad C \overset{!}{=}$ Rand der Kreisscheibe $K_R \perp \vec{a}$, Zentr. O,
Radius R

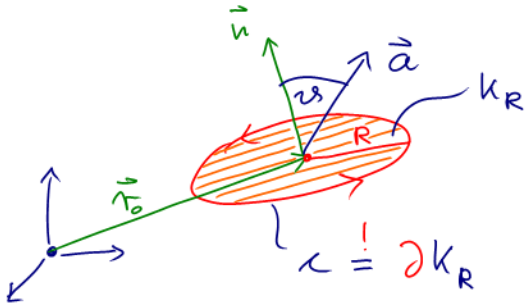


$$\int_{\partial K_R} \vec{A}\, d\vec{l} \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \int_{K_R} \operatorname{rot}\vec{A}\, d\vec{f} \underset{\substack{\operatorname{rot}\vec{A} = 2\vec{a} \parallel d\vec{f} \\ |K_R| = \pi R^2}}{=} \pi R^2 \cdot 2|\vec{a}| \quad \checkmark$$

b) $A(\vec r) = \vec a \times \vec r$, $\longrightarrow \ \operatorname{rot} \vec A = 2\vec a$



$$\int_c \vec A \, d\vec l = \ ?$$



$$\int_c \vec A \, d\vec l = \int_{\partial K_R} \vec A \, d\vec l \overset{\text{Stokes}}{=} \int_{K_R} \operatorname{rot} \vec A \, d\vec f$$

$$= \pi R^2 \, 2 |\vec a| \cos\vartheta \quad !$$

$|K_R| = \pi R^2$

$\operatorname{rot} \vec A = 2\vec a$

$\langle \hat n, \hat a \rangle = \cos\vartheta$

2) Erinnerung:

Vf. $\vec{A} : \mathbb{R}^3 \to \mathbb{R}^3$, folgende Aussagen sind äquivalent:

(i) $\vec{A}$ konservativ

(ii) für jeden geschlossenen Weg c ist $\int_c \vec{A}\, d\vec{l} = 0$

(iii) $\frac{\partial A_i}{\partial x_j} \overset{!}{=} \frac{\partial A_j}{\partial x_i}$

offenbar äquivalent!

(iii') $\operatorname{rot} \vec{A} \overset{!}{=} \vec{0}$

schon gezeigt: (i) $\Leftrightarrow$ (ii), (i) $\Rightarrow$ (iii)

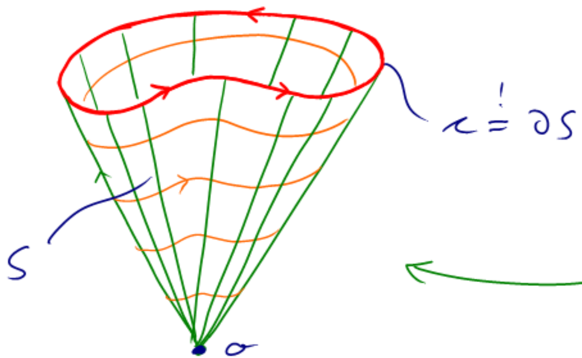
zeigen hier: (iii') $\Rightarrow$ (ii)

Sei c ein beliebiger geschlossener Weg im $\mathbb{R}^3$ und es gelte $\operatorname{rot}\vec{A} = \vec{0}$;

z.z.: $$\int_c \vec{A}\, d\vec{\ell} \overset{!}{=} 0$$

$\boxed{c \text{ geschlossen} \overset{!}{\rightsquigarrow} \text{es gibt ein Flächenstück } S \text{ derart, dass } \partial S \overset{!}{=} c}$

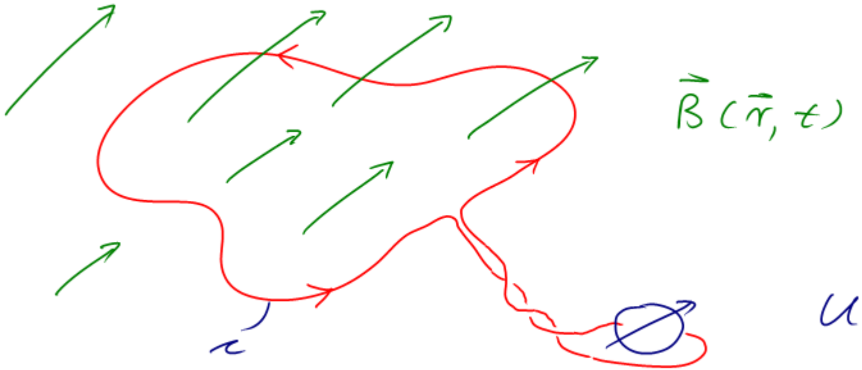
$$c : [u_1, u_2] \longrightarrow \mathbb{R}^3$$
$$u \longmapsto \vec{c}(u)$$



$$S : [u_1, u_2] \times [0,1] \longrightarrow \mathbb{R}^3$$
$$(u, v) \longmapsto \vec{S}(u,v) = v\, \vec{c}(u)$$

Stokes: $$\int_c \vec{A}\, d\vec{\ell} = \int_{\partial S} \vec{A}\, d\vec{\ell} \overset{!}{=} \int_S \operatorname{rot}\vec{A}\, d\vec{f} \overset{\operatorname{rot}\vec{A} = \vec{0}}{=} 0$$

3) Induktionsspannung in einer Leiterschleife

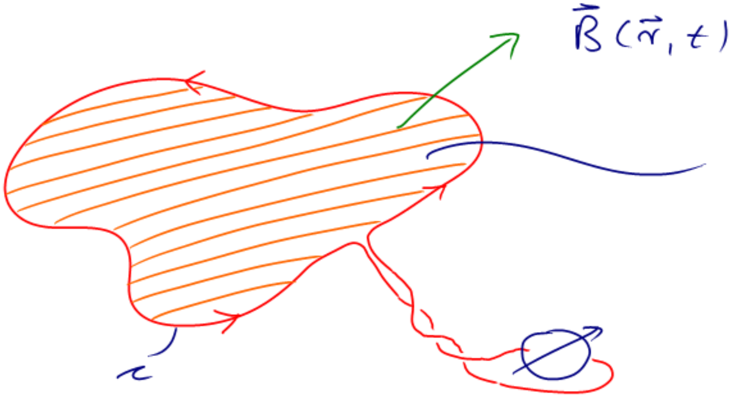


$U(t) = ?$

(i) $$U \overset{!}{=} \int_{\mathcal{C}} \vec{E}\, d\vec{\ell}$$

(ii) Faradaysche Induktionsgesetz:

$$\mathrm{rot}\, \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$



$$S: \quad \partial S \overset{!}{=} c$$

$$\rightsquigarrow \quad \underline{U} = \int_c \vec{E}\, d\vec{\ell} = \int_{\partial S} \vec{E}\, d\vec{\ell} \overset{\text{Stokes}}{=} \int_S \text{rot}\, \vec{E}\, d\vec{f} \overset{\text{Faraday}}{=} -\int_S \frac{\partial \vec{B}}{\partial t}\, d\vec{f}$$

$$\underset{S \text{ statisch}}{=} -\frac{d}{dt} \int_S \vec{B}\, d\vec{f} \quad ;$$

magnet. Fluss durch S : $\Phi(t) := \int_S \vec{B}(t)\, d\vec{f}$

$$\rightsquigarrow \quad \boxed{U = -\frac{d}{dt} \Phi}$$