Integralsätze:

Gauß:

$$\int_{\partial V} \vec{A}\, d\vec{f} = \int_{V} \operatorname{div} \vec{A}\, dV$$

∂V – Fläche, V – Volumen

Stokes:

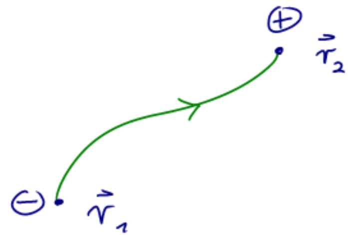
$$\int_{\partial S} \vec{A}\, d\vec{\ell} = \int_{S} \operatorname{rot} \vec{A}\, d\vec{f}$$

∂S – Weg, S – Fläche

HDI:

$$\sum_{\partial c} \Phi = \int_{c} \operatorname{grad} \Phi\, d\vec{\ell}$$

∂c – Punkte, c – Weg



- $\partial c = \{ (\vec{r}_1, -), (\vec{r}_2, +) \}$: Randpunkte des Wegs c
- $\sum_{\partial c} \Phi = -\Phi(\vec{r}_1) + \Phi(\vec{r}_2)$

Integralsätze von Gauß, Stokes und HDI sind Spezialfälle des allgemeinen Satz von Stokes:

$$\boxed{\int_{\partial M} \omega \overset{!}{=} \int_{M} d\omega}$$

M : k-dimensionale Mannigfaltigkeit

ω : $(k-1)$-dimensionale Differenzialform

d : äußere Ableitung

($\longrightarrow$ Analysis III, evtl. Vektoranalysis / Lin. Alg)
SS 21

Gradient und Divergenz in Zylinder- und Kugelkoordinaten

- **Gradient in Zylinderkoordinaten:** $\mathbb{R}_+ \times [0, 2\pi[\times \mathbb{R} \longrightarrow \mathbb{R}^3$

$$(\varrho, \varphi, z) \mapsto \vec{r}(\varrho, \varphi, z) = \begin{pmatrix} \varrho \cos\varphi \\ \varrho \sin\varphi \\ z \end{pmatrix}$$

$\longrightarrow$ lokale ONB:

$$\vec{e}_\varrho = \frac{\partial \vec{r}}{\partial \varrho} \; ; \quad \vec{e}_\varphi = \underline{\underline{\frac{1}{\varrho}}} \frac{\partial \vec{r}}{\partial \varphi} \; ; \quad \vec{e}_z = \frac{\partial \vec{r}}{\partial z} \qquad (1)$$

Skalarfeld $\quad f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$\vec{r} \mapsto f(\vec{r}) \qquad\qquad \vec{r} = (x_1, x_2, x_3)$$

Skalarfeld f in Zylinderkoord.: $\quad \underline{\underline{\tilde{f}}} : \mathbb{R}_+ \times [0, 2\pi[\times \mathbb{R} \longrightarrow \mathbb{R}$

$$(\varrho, \varphi, z) \mapsto \underline{\underline{\tilde{f}(\varrho, \varphi, z) := f(\vec{r}(\varrho, \varphi, z))}}$$

(oft: $\quad \tilde{f}(\varrho, \varphi, z) \equiv f(\varrho, \varphi, z) \leftarrow$ ohne "~")

Gradient von f: $\quad \operatorname{grad} f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\vec{r} \longmapsto \operatorname{grad} f(\vec{r}) = \sum_i \frac{\partial f}{\partial x_i} \underline{\underline{\vec{e}_i}}$$

<u>Gradient von f in Zylinderkoordinaten:</u>

$$\widetilde{\operatorname{grad}}\, f : \mathbb{R}_+ \times [0, 2\pi[\times \mathbb{R} \longrightarrow \mathbb{R}^3$$

$$(\rho, \varphi, z) \longmapsto \widetilde{\operatorname{grad}}\, f(\rho, \varphi, z) := \operatorname{grad} f(\underline{\vec{r}(\rho, \varphi, z)})$$

$$\overset{!}{=} a_\rho \underline{\underline{\vec{e}_\rho}} + a_\varphi \underline{\underline{\vec{e}_\varphi}} + a_z \underline{\underline{\vec{e}_z}}$$

<u>Aufgabe</u>: bestimme die Koeffizienten a_ρ, a_φ, a_z !

$$\bullet \quad \frac{\partial \tilde{f}}{\partial \rho} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \rho} = \Big\langle \widetilde{\operatorname{grad} f} \, , \underbrace{\frac{\partial \vec{r}}{\partial \rho}}_{= \vec{e}_\rho} \Big\rangle = a_\rho$$

d.h. $\boxed{a_\rho = \frac{\partial \tilde{f}}{\partial \rho}}$

$$\bullet \quad \frac{\partial \tilde{f}}{\partial \varphi} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \varphi} = \Big\langle \widetilde{\operatorname{grad} f} \, , \underbrace{\frac{\partial \vec{r}}{\partial \varphi}}_{= \rho \, \vec{e}_\varphi} \Big\rangle = \rho \, a_\varphi$$

d.h. $\boxed{a_\varphi = \frac{1}{\rho} \frac{\partial \tilde{f}}{\partial \varphi}}$

$$\bullet \quad \frac{\partial \tilde{f}}{\partial z} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial z} = \Big\langle \widetilde{\operatorname{grad} f} \, , \underbrace{\frac{\partial \vec{r}}{\partial z}}_{= \vec{e}_z} \Big\rangle = a_z$$

d.h. $\boxed{a_z = \frac{\partial \tilde{f}}{\partial z}}$

→ Gradient in Zylinderkoordinaten:

$$\boxed{\widetilde{\operatorname{grad} f} = \frac{\partial \tilde{f}}{\partial \rho}\,\vec{e}_\rho + \frac{1}{\rho}\frac{\partial \tilde{f}}{\partial \varphi}\,\vec{e}_\varphi + \frac{\partial \tilde{f}}{\partial z}\,\vec{e}_z}$$

Beispiel:

$$\tilde{f}(\rho, \varphi, z) = \rho^2 \cdot \sin\varphi \cdot z$$

$$\rightarrow \quad \operatorname{grad} \tilde{f}(\rho, \varphi, z) = 2\rho \sin\varphi \, z \, \vec{e}_\rho + \rho \cos\varphi \, z \, \vec{e}_\varphi + \rho^2 \sin\varphi \, \vec{e}_z$$

- Gradient in Kugelkoordinaten:

$$(r, \vartheta, \varphi) \longmapsto \vec{r}(r, \vartheta, \varphi) = r \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$\rightarrow$ lokale ONB: $\quad \vec{e}_r = \frac{\partial \vec{r}}{\partial r}, \quad \vec{e}_\vartheta = \frac{1}{r} \frac{\partial \vec{r}}{\partial \vartheta}, \quad \vec{e}_\varphi = \frac{1}{r \sin\vartheta} \frac{\partial \vec{r}}{\partial \varphi}$

genau wie oben folgt:

$$\boxed{\widetilde{\text{grad} f} = \frac{\partial \tilde{f}}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \tilde{f}}{\partial \vartheta} \vec{e}_\vartheta + \frac{1}{r \sin\vartheta} \frac{\partial \tilde{f}}{\partial \varphi} \vec{e}_\varphi}$$

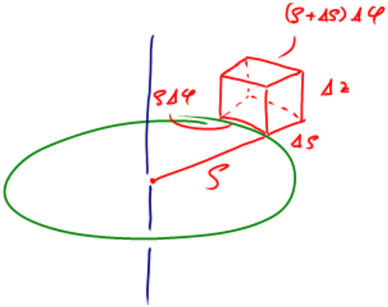
Beispiele:

- $\tilde{f}(r, \vartheta, \varphi) = r^2 \sin\vartheta \quad \rightarrow \quad \widetilde{\text{grad} f} = 2 r \sin\vartheta \, \vec{e}_r + r \cos\vartheta$
- $\tilde{f}(r, \cancel{\vartheta}, \cancel{\varphi}) = h(r) \quad \longrightarrow \quad \widetilde{\text{grad} f} = h'(r) \, \vec{e}_r$

• Divergenz in Zylinderkoordinaten:

Vektorfeld $\vec{A}$: $\vec{A}(\rho, \varphi, z) = \tilde{A}_\rho \, \vec{e}_\rho + \tilde{A}_\varphi \, \vec{e}_\varphi + \tilde{A}_z \, \vec{e}_z$

bestimme Divergenz mittels $\operatorname{div} \vec{A} = \lim\limits_{|V| \to 0} \frac{1}{|V|} \int\limits_{\partial V} \vec{A} \, d\vec{f}$:



$\rightarrow \quad |\Delta V| = \rho \, \Delta\rho \, \Delta\varphi \, \Delta z$

$$\rightarrow \int\limits_{\partial V} \vec{A} \, d\vec{f} = \overbrace{\tilde{A}_\rho(\rho+\Delta\rho, \varphi, z)(\rho+\Delta\rho)\,\Delta\varphi\,\Delta z - \tilde{A}_\rho(\rho, \varphi, z)\,\rho\,\Delta\varphi\,\Delta z}^{= \frac{\partial}{\partial\rho}(\tilde{A}_\rho \cdot \rho)\,\Delta\rho\,\Delta\varphi\,\Delta z}$$

$$+ \tilde{A}_\varphi(\rho, \varphi+\Delta\varphi, z)\,\Delta\rho\,\Delta z - \tilde{A}_\varphi(\rho, \varphi, z)\,\Delta\rho\,\Delta z$$

$$+ \tilde{A}_z(\rho, \varphi, z+\Delta z)\,\rho\,\Delta\rho\,\Delta\varphi - \tilde{A}_z(\rho, \varphi, z)\,\rho\,\Delta\rho\,\Delta\varphi$$

$\rightarrow$

Divergenz in Zylinderkoordinaten:

$$\boxed{\operatorname{div} \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \tilde{A}_\rho \right) + \frac{1}{\rho} \frac{\partial \tilde{A}_\varphi}{\partial \varphi} + \frac{\partial \tilde{A}_z}{\partial z}}$$

analog: Divergenz in Kugelkoordinaten:

$$\boxed{\operatorname{div} \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \tilde{A}_r \right) + \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta} \left(\sin\vartheta \, \tilde{A}_\vartheta \right) + \frac{1}{r \sin\vartheta} \frac{\partial \tilde{A}_\varphi}{\partial \varphi}}$$

→ Beispiele:

- $\operatorname{div} \vec{r} = \operatorname{div}(r \vec{e}_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 r) = 3$ ✓
- $\operatorname{div} \frac{\hat{r}}{r^2} = \operatorname{div} \frac{\vec{e}_r}{r^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r^2} \right) = 0$ ✓